A Multiple-Risk Optimal Hedging Model: An Example

by

Gboroton F. Sarassoro and Raymond M. Leuthold

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Introduction

The liberalization of trade among countries, the increased mobility of capital, and the introduction of a flexible exchange rate regime have created an integrated world economy where domestic public policies have global impacts and disturbances in one country quickly transmit to others. Multinational firms and export agencies take advantage of a greater pool of investment opportunities and larger markets respectively. However, they also face greater economic risks resulting from variability in price, exchange rate and/or interest rate. They are also exposed to political risk.

Several theories have attempted to explain and predict exchange and interest rate differentials across countries. The Purchasing Power Parity (PPP) theorem states that a change in the exchange rate differentials between two countries is met with a corresponding change in price differentials, resulting in no revenue loss or gain to importers or exporters. Similarly, international economic theory links interest and exchange rates through the Interest Rate Parity (IRP) theorem. The Covered Interest Rate Parity (CIP) hypothesis states that for countries which allow free movement of capital, the interest rate differential will equal the differential between the spot and forward exchange rates. The Uncovered Interest Rate Parity (UIP) hypothesis attempts to explain the behavior of speculators. Some argue that when the CIP holds, speculators will take open positions in the forward exchange markets as long as expected and spot exchange rates offer them an expected profit. Empirical evidence refutes the UIP and the PPP hypotheses; only the CIP seems to hold true for major currencies (Officer and Willet, 1970; Otani and Tiwari, 1981; Solnik, 1978; Isard, 1987). The implication of this empirical evidence is that contrary to traditional international economic theory, multinational firms and international traders faced with price, currency and/or interest rate risks are justified in taking steps to manage them.

The objective of this paper is to develop a risk management strategy for export agencies exposed to price, currency and interest rate risks. In many cases, these agencies export commodities whose prices are quoted in a foreign currency and fluctuate widely due to changes in supply and demand conditions. In addition, these agencies may have to invest their excess liquidity in financial markets where interest rates are variable.

Specifically, this paper utilizes portfolio theory to demonstrate how traders can use the futures market to manage multiple risks simultaneously. First, we derive a general portfolio model for optimal futures market hedging within the context of the quantity, price, interest and exchange rate risks faced by Cote D'Ivoire in their exporting of cocoa and coffee. Second, we present the

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estimating procedure and optimal commodity and financial futures positions for the Cote D'Ivoire export agency, Caisse. Finally, we discuss the implications of this research. The approach developed here is appropriate to many international trading situations.

Theoretical Background and Model

Markowitz (1952) portfolio theory as applied by Johnson (1960) and Stein (1961) is the basis for our risk management model. Several authors have used the Johnson-Stein model to show the empirical employment of the futures markets in managing price risk (Peck, 1975), price and quantity risks (Rolfo, 1980), exchange rate risk (Soenen, 1979) and interest rate risk (Franckle, 1980). Recently, Thompson and Bond (1987) examined the management of price and exchange rate risks simultaneously. This study builds on these previous studies in developing a hedging model.

Using the mean-variance framework, it is assumed that the objective of Caisse is to maximize expected revenue subject to a certain level of risk where risk is measured by the variance of revenue.

The objective function to be maximized can be formalized as:

$$\Omega = E_t(Y_{t+1}) - \delta \ VAR_t(Y_{t+1}) \qquad (1)$$

where,

Y_{t+1} is the revenue in period t+1,
E_t is the expectation operator,
VAR_t is the variance operator, and,
δ is the risk aversion parameter ($\delta>0$).

The revenue in t+1 is a function of the actions undertaken by Caisse before harvest (t-1) and at harvest (t) both in the cash and the futures markets. At harvest, Caisse through private exporters buys quantities Q_{cc} and Q_{ce} of cocoa and coffee at fixed domestic prices (CFA francs) $P_{cc}{}^d$ and $P_{ce}{}^d$ respectively. These quantities are sold in the world markets (in dollars) at prices P_{cc} and P_{ce} respectively. The proceeds are then invested at a one-period rate of interest r. The total revenue of the cash cocoa and coffee activities (after converting dollars to CFA francs by exchange rate X) are designated as R_{cc} and R_{ce} respectively.

Before harvest Caisse sells quantity H_{cc} and H_{ce} of cocoa and coffee forward in the futures markets at price $f_{cc}{}^{t-1}$ and $f_{ce}{}^{t-1}$, respectively. At harvest, Caisse buys back these quantities at price $f_{cc}{}^t$ and $f_{ce}{}^t$ respectively. In local currency, the return from these activities in the cocoa and coffee futures markets are $R_{cc}{}^f$ and $R_{ce}{}^f$, respectively.

Before harvest, Caisse buys quantity C of foreign currency futures at price $X_f{}^{t-1}$ to be sold back at time t+1 at $X_f{}^{t+1}$. The return from this action is $R_c{}^f$.

Before harvest Caisse buys quantity I of interest rate futures contracts paying a rate $r_f{}^{t-1}$ to be sold back at harvest (t) at rate $r_f{}^t$. The return from this investment is $R_r{}^f$.

In the risk management context the unknowns to be solved for by Caisse are the levels of the commodity hedges, H_{cc} and H_{ce}, the amount of currency hedge, C, and the level of interest rate hedge, I.

The net income generated by Caisse is:

$$Y_{t+1} = K'R \qquad (2)$$

where,
K is a (6X1) vector of ones and futures positions, $K'=[1\ 1\ H_{cc}\ H_{ce}\ C\ I]$, and,
R is a (6X1) vector of cash and futures returns, $R'=[R_{cc}\ R_{ce}\ R_{cc}^{f}\ R_{ce}^{f}\ R_{c}^{f}\ R_{r}^{f}]$ with the returns defined as:

$$R_{cc} = [(1+r_t)P_{cct}X_{t+1} - P_{cc}^{d}]Q_{cc},$$

$$R_{ce} = [(1+r_t)P_{cet}X_{t+1} - P_{ce}^{d}]Q_{ce},$$

$$R_{cc}^{f} = (f_{cc}^{t-1} - f_{cc}^{t})X_t,$$

$$R_{ce}^{f} = (f_{ce}^{t-1} - f_{cc}^{t})X_t,$$

$$R_{c}^{f} = (X_f^{t-1} - X_f^{t+1}),$$

$$R_{r}^{f} = (r_f^{t-1} - r_f^{t}).$$

The objective function (1) becomes:

$$\Omega = K'\ E(R) - \delta\ [K'RR'K].$$

The above objective function is concave, consequently the maximum is obtained when the first derivatives with respect to the decision variables H_{cc}, H_{ce}, C, and I are equal to zero. That is:

$$\frac{d(\Omega)}{d\ K_1} = E(R_1) - \delta\ VV'K = 0 \qquad (3)$$

where, $K_1 = [H_{cc}\ H_{ce}\ C\ I]$, $R'_1=[R_{cc}^{f}\ R_{ce}^{f}\ R_{c}^{f}\ R_{r}^{f}]$ and VV' is the following (4X6) matrix:
$VV' = [COV(i,j)]$, with $i = R_{cc}^{f}, R_{ce}^{f}, R_{c}^{f}, R_{r}^{f}$ and $j = R_{cc}\ R_{ce}\ R_{cc}^{f}\ R_{ce}^{f}\ R_{c}^{f}$ and R_{r}^{f}.

Premultiplying equation (3) by the following (4X4) matrix Z,

$$Z = \begin{bmatrix} VAR^{-1}(R_{cc}^{f}) & 0 & 0 & 0 \\ 0 & VAR^{-1}(R_{ce}^{f}) & 0 & 0 \\ 0 & 0 & Var^{-1}(R_{c}^{f}) & 0 \\ 0 & 0 & 0 & Var^{-1}(R_{r}^{f}) \end{bmatrix}$$

leads to

$1/\delta \; ZE(R_1) - ZVV'K = 0$ (4)

or,

$1/\delta \; S - PK = 0$

where $P = ZVV'$ is a (4X6) matrix whose elements are of the form $COV(R_i, R_j)/VAR(R_j)$, or simple regression coefficients, and, $S = ZE(R_1)$ is a (4X1) vector of ratios of expected return to variance.

In order to isolate the four unknown elements of K (i.e., H_{cc}, H_{ce}, C and I), matrices P and K are written in the following partioned forms:

$P = (\Sigma_2 | \Sigma_4)$

and,

$K' = (1 \; 1 \; | \; H_{cc} \; H_{ce} \; C \; I)$

where,

$\Sigma_2 = [a_{ij}]$ with i=3,6 j=1,2

and,

$\Sigma_4 = [a_{ij}]$ with i=3,6 j=3,6.

Define a_{ij} as the coefficients of simple regressions. Letting i=j=1,6 refer to R_{cc}, R_{ce}, R_{cc}^f, R_{ce}^f, R_c^f and R_r^f respectively, a_{ij} is the coefficient of the simple regression of i on j.

Equation (4) can now be expressed as:

$$1/\delta \; S - \Sigma_2 I_2 - \Sigma_4 K_1 = 0 \qquad (5)$$

where $I_2' = [1 \; 1]$.

Since the $COV(R_i, R_i)/VAR(R_i) = 1$, the matrix Σ_4 has ones on its principal diagonal. Consequently, it can be written as:

$$\Sigma_4 = \Sigma_4^1 + I \qquad (6)$$

where I is the identity matrix and Σ_4^1 is the same matrix as Σ_4 except it has zeros on its first diagonal instead of ones. Substituting (6) into (5) the solution of the hedging problem becomes:

$$K_1 = 1/\delta \; S - \Sigma_2 I_2 - \Sigma_4^1 K_1 \qquad (7)$$

or

$$K_1^* = \Sigma_4^{-1}(1/\delta \; S - \Sigma_2 I_2). \qquad (8)$$

Appendix 1 gives the detailed expression of the optimal hedges. As indicated in previous hedging literature (Rolfo, 1980; Anderson and Danthine, 1980), the optimal futures positions K_1^* includes a speculative $1/\delta \Sigma_4^{-1}$ S and a hedge component $-\Sigma_4^{-1}\Sigma_2 I_2$. For a highly risk averse decision maker (δ goes to infinity) or risky futures markets (S approaches zero) the optimal hedge coincides with the risk minimization strategy and $K_1^* = -\Sigma_4^{-1}\Sigma_2 I_2$.

We can use equation (8) to estimate the optimal hedges H_{cc}, H_{ce}, C and I for selected values of the risk parameter δ in the interval from 10^{-5} for very low risk aversion to 10^5 for very high risk aversion.

The hedging strategy developed here will be evaluated by comparing it to the no hedge strategy. Since δ is assumed fixed in equation (1), the ratio of the expected income to the variance can be used to compare the two strategies. For large values of the risk aversion parameter, δ, the hedging effectiveness coefficient $e = 1 - VAR(Y_h)/VAR(Y_u)$ can be used to compare the hedging and no hedging strategies.[1] $VAR(Y_h)$ and $VAR(Y_u)$ are the variances of the revenue associated with the hedged and the unhedged portfolios respectively.

Empirical Findings

Cocoa and coffee are harvested in Cote D'Ivoire continuously from October up to September of the following year with the bulk of the harvest occurring from December to March. In this paper, the cocoa and coffee seasons are divided into the following four periods: October-December, January-March, April-June and July-September.

The optimal hedging strategy proposed here assumes that just before each period (in September, December, March and June), Caisse takes a position in the cocoa, coffee, currency and interest rate futures markets to be reversed at the end of the period (December, March, June and September) when the cash commodities are sold in the spot markets and the proceeds are invested.[2] The contracts used for each quarter are December, March, June and September. However, since there are no cocoa and coffee futures contracts in June, the July futures contract is used instead for those two commodities. Also, there is no futures market for the CFA franc, and the French franc, to which the CFA franc is tied (1FF = .02 CFA franc), has no active futures market. Consequently, an alternative futures currency, the British pound, is used to cross hedge the CFA franc. The return from the futures currency market R_c^f becomes $R^f_c = (BP^{t-1} - BP^{t+1})Q_{t+1}$ where BP is the British pound futures price in U.S. dollars and Q is the British pound/U.S. dollar exchange rate. This adjustment allows the optimal hedge C to be interpreted as CFA francs.

The basic data needed to calculate the commodity and financial optimal hedges are the monthly average futures prices for those months futures contracts bought and sold and the total quantities of cocoa and coffee exported by Cote D'Ivoire during the quarter. The proceeds are invested in 3-month U.S. T-bills. Also needed are cocoa and coffee cash prices for the months of December, March, June and September. Finally, the domestic prices for cocoa and coffee are fixed for each year. The analysis of this paper covers the period from 1976 to 1986.

Expectations about futures returns in the commodity and financial futures markets are needed to estimate the optimal hedges. Most hedgers presumably are also profit seekers (Working, 1953) who make use of all the available information in forecasting futures returns. One convenient way to take into account all the

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[1] Large values of S correspond to the situation when the speculative component is zero and the optimal hedge is the same as risk minimization.

[2] The currency position is held for two quarters. Consequently, Caisse reverses its initial currency positions taken in September, December, March and June at the end of March, June, September and December respectively.

available information is to use time series techniques (Bond, et al., 1985) which by design assures that the forecast residuals are white noise (i.e., expected value is zero). Consider the following expectation models:

$$R_i^{t+1} = E_t(R_i^{t+1}) + U_{t+1}$$

$$R_j^{t+1} = E_t(R_j^{t+1}) + V_{t+1}$$

where U and V are white noise residual terms (i.e; E(U) = 0 and E(V)=0). The variance of R_i and R_j are:

$$VAR_t(R_i) = VAR_t(U)$$

$$VAR_t(R_j) = VAR_t(V)$$

respectively.

The covariance between R_i and R_j is:

$$COV_t(R_i,R_j) = COV_t(U,V).$$

Thus, the parameters of equation (8) which are of the form $COV(R_i,R_j)/VAR(R_j)$ can be estimated by the coefficient of the regression U on V since $COV(R_i,R_j)/VAR(R_j) = COV(U,V)/VAR(V)$.

In this paper due to the limited number of observations, an expectation model based on a time series technique may not lead to white noise residuals. Consequently, it is assumed that the best forecasts of the future returns are the average of past returns. The parameters of the hedge component in equation (8) are estimated by simple regression on returns instead of forecast errors.

Table 1 gives the ratios of the futures returns to their respective variances for the four periods being studied. All but one ratio are less than absolute one, and in most cases these ratios are very close to zero, suggesting that little speculative opportunity exists.

Actual estimation of the optimal hedge for values of δ varying from 10^{-5} to 10^{5} show that the optimal hedges do not vary significantly. Consequently, only the results corresponding to the risk minimization strategy ($\delta = 10^{-5}$) are reported. The hedging model can therefore be evaluated using the hedging efficiency criteria ($e = 1 - Var(Y_h)/Var(Y_u)$).

Table 2 reports the optimal cocoa, coffee, currency and interest rate hedge ratios and the proportion of total risk eliminated by the hedging strategy. Normally, Caisse would sell cocoa and coffee futures and buy currency and interest rate futures. These normal positions are represented by positive signs in table 2 while opposite positions are noted with negative signs. The hedging ratios range from buying futures contracts larger than the cash position (currency in September-March), to very small futures positions (cocoa in September-December), to buying more futures contracts than the size of the cash position when selling contracts is the normal position (coffee in March-June). Most importantly, the hedging effectiveness column indicates that the commodity

Table 1 The Ratios of the Expected Futures Returns to Variances for the Cocoa, Coffee, Currency and Interest Rate Futures Markets

Periods	Cocoa	Coffee	Currency[a]	Interest Rate
September-December	$1.9x10^{-6}$	$-2.94x10^{-6}$	-.00036	-0.007
December-March	$-16x10^{-7}$	$2.0x10^{-7}$	-1.97	-0.28
March-June	$-1.25x10^{-7}$	$1.53x10^{-7}$	0.65	0.75
June-September	$1x10^{-6}$	$-11x10^{-7}$	-0.39	-0.004

[a] The currency hedges span two periods, not one (see text).

Table 2 Optimal Commodity and Financial Hedge Ratios, and Hedging Effectiveness (in percent)

	Optimal Hedges				
Periods	Cocoa	Coffee	Currency[a]	Interest Rate	Hedging Effectiveness
September-December	-5	190	339	46	60
December-March	17	-40	-147	-92	27
March-June	6	-314	-15	51	89
June-September	50	22	8	8	42

[a] The currency hedges span two periods, not one (see text).

and financial futures markets are useful risk management tools since Caisse can reduce between 27 and 89 percent of the risk associated with cocoa and coffee export revenue. These represent substantial reduction in risks.

Several observations can be drawn from the above results. First, across the four quarters the average risk reduction is 54 percent, which is noteworthy. The results also obtained in this analysis suggest that contrary to traditional hedging theory, taking futures positions greater than the cash position may be consistent with optimal hedging decisions. Sometimes these results show the futures position to be on the same side, rather than the opposite side, of the cash position. Gemmill (1980) found previously that long positions in cocoa futures were consistent with risk minimization. Similarly, Rolfo (1980) found for low risk averse producers that a long position in cocoa futures was optimal. However, both of those studies involved single commodity hedges.

Second, there is considerable variability in the hedge positions from quarter to quarter. This means a government agency needs to be flexible in establishing futures positions and be willing to alter them in subsequent quarters. This need for flexibility may suggest why many stabilization schemes fail where countries are locked into specific scenarios for a whole year.

Third, some of the results of this paper may be difficult to implement because political leaders in many developing countries believe that participation in futures markets is speculation. Therefore, it may be difficult to convince the decision makers at Caisse to take futures positions at all, let alone futures positions which are greater than the cash positions. Consequently, upper and lower bounds corresponding to the expected cash position and zero respectively, can be put on the different hedges. For example, when the hedge is found to be negative, set it equal to zero. Similarly, a futures position greater than the corresponding expected cash position is set to that expected cash position. Experimentation with this constrained risk minimization strategy reduced the hedging effectiveness for all four periods on average from 54 percent to 21 percent, making these results less appealing. These results are also nonoptimal.

Fourth, data limitations affect the results. We were constrained to quarterly data, but certainly Caisse makes decisions more frequently. Presumably, Caisse has additional data for analysis that would conform more closely with their decision horizons.

Conclusion

This paper demonstrates how a marketing agency faced with multiple international risks (quantity, price, exchange rate and interest rate) may use commodity and financial futures markets as management tools. In particular, this paper develops an optimal hedging model, and applies it to Cote D'Ivoire cocoa and coffee exports.

When the average of past returns is used as an expectation model, the cocoa, coffee, British pound and 3-month U.S. T-bill futures markets offer little speculative opportunity. However, when the objective of the decision maker is to minimize risk, the cocoa, coffee, exchange rate and interest rate futures markets provide substantial risk reduction opportunity. In particular, it was found that the government export agency of Cote D'Ivoire may reduce from 27 to 89 percent of the risk it faces in marketing cocoa and coffee.

Finally, it was found that on occasions risk minimization would require futures positions greater than the expected cash positions. Such policy may be politically difficult to implement in most developing countries. However, this study demonstrates the potential for substantial risk reduction when managing multiple risks simultaneously.

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APPENDIX 1

Derivation of the Optimal Hedges

The objective function to be maximized is:

$$E(Y) - 1/2\ \delta VAR(Y) \qquad A1$$

where,

$Y = R_{cc} + R_{ce} + H_{cc}\ R^f_{cc} + H_{ce}\ R^f_{ce} + C\ R^f_c + I\ R^f_r$,

δ is the risk parameter coefficient,

E(.) is the expectation operator and,

VAR(.) is the variance operator.

The detailed expression of the variance of Y is:

$$VAR(Y) = VAR(R_{cc}) + VAR(R_{ce}) + H_{cc}^2\ VAR(R^f_{cc}) + H_{ce}^2\ VAR(R^f_{ce}) + C^2\ VAR(R^f_c) + I^2\ VAR(R^f_r) + 2(COV(R_{cc},R_{ce}) + H_{cc}\ COV(R_{cc},R^f_{cc}) + H_{ce}\ COV(R_{cc},R^f_{ce}) + C\ COV(R_{cc},R^f_c) + I\ COV(R_{cc},R^f_r) + H_{cc}\ COV(R_{ce},R^f_{cc}) + H_{ce}\ COV(R_{ce},R^f_{ce}) + C\ COV(R_{ce},R^f_c) + I\ COV(R_{ce},R^f_r) + H_{cc}H_{ce}\ COV(R^f_{cc},R^f_{ce}) + H_{cc}\ C\ COV(R^f_{cc},R^f_c) + H_{cc}\ I\ COV(R^f_{cc},R^f_r) + H_{ce}\ C\ COV(R^f_{ce},R^f_c) + H_{ce}\ I\ COV(R^f_{ce},R^f_r) + C\ I\ COV(R^f_c,R^f_r).$$

The concavity of the objective function assures that the maximum is obtained at the point where the first derivatives of A1 with respect to the decision variables H_{cc}, H_{ce}, C and I are equal to zero. These first derivatives are:

$$\frac{d\ E(U(Y))}{d\ H_{cc}} = E(R^f_{cc}) - \delta\ [H_{cc}\ VAR(R^f_{cc}) + COV(R_{cc},R^f_{cc}) + COV(R_{ce},R^f_{cc}) + H_{ce}\ COV(R^f_{cc},R^f_{ce}) + C\ COV(R^f_{cc},R^f_c) + I\ COV(R^f_{cc},R^f_r)]=0 \qquad A2$$

$$\frac{d\ E(U(Y))}{d\ H_{ce}} = E(R^f_{ce}) - \delta\ [H_{ce}\ VAR(R^f_{ce}) + COV(R_{cc},R^f_{ce}) + COV(R_{ce},R^f_{ce}) + H_{cc}\ COV(R^f_{cc},R^f_{ce}) + C\ COV(R^f_c,R^f_{ce}) + I\ COV(R^f_r,R^f_{ce})]=0 \qquad A3$$

$$\frac{d\ E(U(Y))}{d\ C} = E(R^f_c) - \delta\ [C\ VAR(R^f_c) + COV(R_{cc},R^f_c) + COV(R_{ce},R^f_c) + H_{cc}\ COV(R^f_{cc},R^f_c) + H_{ce}\ COV(R^f_{ce},R^f_c) + I\ COV(R^f_r,R^f_c)]=0 \qquad A4$$

$$\frac{d\,E(U(Y))}{d\,I} = E(R^f_r) - \delta\,[I\ VAR(R^f_r) + COV(R_{cc},R^f_r) + COV(R_{ce},R^f_r) + H_{cc}\ COV(R^f_{cc},R^f_r) + H_{ce}\ COV(R^f_{ce},R^f_r) + C\ COV(R^f_c,R^f_r)]=0 \qquad A5$$

respectively.

Dividing equations A2, A3, A4 and A5 through by $VAR(R^f_{cc})$, $VAR(R^f_{ce})$, $VAR(R^f_c)$ and $VAR(R^f_r)$, respectively, and solving for H_{cc}, H_{ce}, C and I give the following equations:

$$H_{cc} = \frac{E(R^f_{cc})}{2\delta\ VAR(R^f_{cc})} - \frac{COV(R^f_{cc},R_{cc})}{VAR(R^f_{cc})} - \frac{COV(R^f_{cc},R_{ce})}{VAR(R^f_{cc})} - H_{ce}\frac{COV(R^f_{cc},R^f_{ce})}{VAR(R^f_{cc})} - C\frac{COV(R^f_{cc},R^f_c)}{VAR(R^f_{cc})} - I\frac{COV(R^f_{cc},R^f_r)}{VAR(R^f_{cc})}$$

$$H_{ce} = \frac{E(R^f_{ce})}{2\delta\ VAR(R^f_{ce})} - \frac{COV(R^f_{ce},R_{cc})}{VAR(R^f_{ce})} - \frac{COV(R^f_{ce},R_{ce})}{VAR(R^f_{ce})} - H_{cc}\frac{COV(R^f_{ce},R^f_{cc})}{VAR(R^f_{ce})} - C\frac{COV(R^f_{ce},R^f_c)}{VAR(R^f_{ce})} - I\frac{COV(R^f_{ce},R^f_r)}{VAR(R^f_{ce})}$$

$$C = \frac{E(R^f_c)}{2\delta\ VAR(R^f_c)} - \frac{COV(R^f_c,R_{cc})}{VAR(R^f_c)} - \frac{COV(R^f_c,R_{ce})}{VAR(R^f_c)} - H_{cc}\frac{COV(R^f_c,R^f_{ce})}{VAR(R^f_c)} - H_{ce}\frac{COV(R^f_c,R^f_{ce})}{VAR(R^f_c)} - I\frac{COV(R^f_c,R^f_r)}{VAR(R^f_c)}$$

$$I = \frac{E(R^f_r)}{2\delta\ VAR(R^f_r)} - \frac{COV(R^f_r,R_{cc})}{VAR(R^f_r)} - \frac{COV(R^f_r,R_{ce})}{VAR(R^f_r)} - H_{cc}\frac{COV(R^f_r,R^f_{ce})}{VAR(R^f_r)} - H_{ce}\frac{COV(R^f_r,R^f_{ce})}{VAR(R^f_r)} - C\frac{COV(R^f_r,R^f_c)}{VAR(R^f_r)}$$

or,

$$H_{cc} = 1/\delta\ S_1 - a_{31} - a_{32} - a_{34}\ H_{ce} - a_{35}\ C - a_{36}\ I$$

$$H_{ce} = 1/\delta\ S_2 - a_{41} - a_{42} - a_{43}\ H_{cc} - a_{45}\ C - a_{46}\ I,$$

$C = 1/\delta\ S_3 - a_{51} - a_{52} - a_{53}\ H_{cc} - a_{54}\ H_{ce} - a_{56}\ I$ and,

$I = 1/\delta\ S_4 - a_{61} - a_{62} - a_{63}\ H_{cc} - a_{64}\ H_{ce} - a_{65}\ C$,

where:
$S_1 = E(R^f{}_{cc})/VAR(R^f{}_{cc})$,

$S_2 = E(R^f{}_{ce})/VAR(R^f{}_{ce})$,

$S_3 = E(R^f{}_{c})/VAR(R^f{}_{c})$,

$S_4 = E(R^f{}_{r})/VAR(R^f{}_{r})$ and,

$1/\delta\ S_1$, a_{31} a_{32}, a_{34}, a_{35} and a_{36} correspond in order with the elements in equation A2,

$1/\delta\ S_2$, a_{41}, a_{42}, a_{43}, a_{45} and a_{46} correspond in order with the elements in equation A3,

$1/\delta\ S_3$, a_{51}, a_{52}, a_{53}, a_{54} and a_{56} correspond in order with the elements in equations A4 and,

$1/\delta\ S_4$, a_{61}, a_{62}, a_{63}, a_{64} and a_{65} correspond in order with the elements in equation A5.

In matrix form the above equations can be written as:
$K_1 = 1/\delta\ S - \Sigma_2\ I_2 + \Sigma_4{}^1\ K_1$,
where,
$K_1' = [H_{cc}\ H_{ce}\ C\ I]$,

$S' = [S_1\ S_2\ S_3\ S_4]$,

$$\Sigma_2 = \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{bmatrix},$$

$$I_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and,

$$\Sigma_4{}^1 = \begin{bmatrix} 0 & a_{34} & a_{35} & a_{36} \\ a_{43} & 0 & a_{45} & a_{46} \\ a_{53} & a_{54} & 0 & a_{56} \\ a_{63} & a_{64} & a_{65} & 0 \end{bmatrix}.$$