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CONDITIONAL HETEROSCEDASTIC ERROR PROCESSES AND THE TIME PATTERN OF OPTIMAL HEDGE RATIOS

Robert J. Myers*

I. Introduction

Hedging on futures markets provides commodity producers and traders with the opportunity to manage risks and make additional profits (or losses). However, futures markets also complicate the decision process. As well as taking out spot positions, hedgers must choose the proportion of spot positions that are covered by opposite positions on the futures market. This latter problem is known as choosing an optimal hedge ratio.

Under certain simplifying assumptions optimal hedge ratios can be characterized by a simple rule -- set the hedge ratio equal to the ratio of the covariance between spot and futures prices to the variance of the futures price (Anderson and Danthine; Benninga, Eldor and Zilcha; Kahl).[1] But to operationalize this simple hedging rule the relevant moments must be estimated using available data. The conventional approach to estimation is to run a simple regression of spot price levels (or changes) on futures price levels (or changes) and use the resulting slope parameter estimate as the recommended hedge ratio. However, Myers and Thompson have shown that this only estimates a ratio of the unconditional covariance between spot and futures prices (or price changes) to the unconditional variance of the futures price (or futures price changes). Since hedging decisions depend on information available at the time the decision is made, a ratio of <u>conditional</u> moments is actually required. As an alternative to simple regression, Myers and Thompson suggested a time series approach which takes account of relevant conditioning information.

A weakness in both the simple regression and time series approaches to optimal hedge ratio estimation is that the conditional covariance between spot and futures prices and the conditional variance of the futures price are both assumed to be constant over time. This necessarily restricts the optimal hedge ratio to be constant as well.[2] Yet commodity prices are clearly more volatile (bigger conditional variance) in some periods than in others. An example is the jump in volatility that occurred during the commodities boom of 1973 (Bosworth and Lawrence). Since the conditional variances and covariances of spot and futures prices appear to change over time, optimal hedge ratios also may change and estimation procedures should attempt to capture this effect.

This paper outlines and applies three models for estimating time-varying optimal hedge ratios. Each model features a conditionally heteroscedastic error process (the conditional covariance matrix of spot and futures prices changes over time). However, the models differ in terms of their generality,

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complexity, and ease of estimation. The first model requires calculating moving sample variances and covariances of previous prediction errors. The model is easy to apply but highly restrictive. The second model is the autoregressive conditional heteroscedastic (ARCH) framework of Engle. Efficient estimation of the ARCH model requires a nonlinear maximum likelihood routine but ordinary least squares (OLS) is a simple alternative. Though not efficient, OLS is consistent and also is straightforward to apply. The third model is the generalized ARCH (GARCH) framework of Bollerslev; also see Engle and Bollerslev; and Baillie and Bollerslev. Nonlinear maximum likelihood is the only available estimation procedure. Both the ARCH and GARCH models have sound foundations in distribution theory but the GARCH model is more general since it provides a parsimonious parameterization of a very wide class of conditional heteroscedastic error processes (Bollerslev).

The next section outlines a model of spot and futures prices with time-varying optimal hedge ratios. Each of the three methods for estimating the time pattern of the conditional covariance matrix is then discussed in the context of this model. Next the methods are applied to estimating the time path of optimal storage hedging for wheat in Michigan. Results vary depending on which method is used but all three methods suggest substantial temporal variation in the optimal hedge ratio.

II. Time-Varying Optimal Hedge Ratios

Realizations of spot and futures prices for a commodity at time t are comprised of two components -- an expectation conditioned on information available at time t-1 and a random shock that is unpredictable. Formally, this can be expressed

$$(1) \qquad p_t = E(p_t|\Omega_{t-1}) + u_t$$

$$(2) \qquad f_t = E(f_t|\Omega_{t-1}) + v_t$$

where: p_t is spot price at t;
f_t is futures price quoted at t for delivery at a future date;
u_t is the prediction error for spot price;
v_t is the prediction error for the futures price; and
Ω_{t-1} is a set of information available at t-1.

The prediction errors are assumed to be serially uncorrelated with expected value zero (conditional on the information set). If futures markets are unbiased and there is no output uncertainty, it is well known that the optimal hedge ratio, r, satisfies[3]

$$r_{t-1} = \frac{Cov(p_t,f_t|\Omega_{t-1})}{Var(f_t|\Omega_{t-1})}$$

or, equivalently

$$(3) \qquad r_{t-1} = \frac{Cov(u_t,v_t|\Omega_{t-1})}{Var(v_t|\Omega_{t-1})}.$$

The conventional approach to estimating optimal hedge ratios is to assume that the covariance matrix of the prediction errors, H, is constant over time:

(4) $E(\varepsilon_t \varepsilon_t' | \Omega_{t-1}) = H$ for all t

where: $\varepsilon_t' = (u_t \; v_t)$.

Next, some simple model is chosen for the conditional means in (1) and (2). For example, it might be assumed that

$$E(p_t | \Omega_{t-1}) = p_{t-1}; \text{ and}$$

$$E(f_t | \Omega_{t-1}) = f_{t-1}.$$

Given this model for the conditional means, it is now possible to observe the prediction errors which are just equal to the period to period price changes:

(5) $u_t = p_t - p_{t-1}$

(6) $v_t = f_t - f_{t-1}$.

Assuming the prediction errors are normally distributed, the maximum likelihood estimator of H given a sample of T observations is

$$\hat{H} = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t \varepsilon_t'.$$

Thus, the estimated optimal hedge ratio is

$$\hat{r}_T = \sum_{t=1}^{T} u_t v_t / \sum_{t=1}^{T} v_t^2.$$

This estimate can be obtained as the slope coefficient from a simple regression of spot price changes on futures price changes. For more general models of the conditional means in (1) and (2) a time series approach to estimation is required (see Myers and Thompson).

Now suppose that the covariance matrix of the prediction errors actually changes over time. Such fluctuations in market volatility might result from changing fundamentals, speculative bubbles, and/or seasonal effects. In this case, the covariance matrix of the prediction errors as defined in (4) must be time subscripted to indicate time variation:

$$E(\varepsilon_t \varepsilon_t' | \Omega_{t-1}) = H_t$$

Furthermore, since the covariance matrix changes over time, so will the optimal hedge ratio defined by (3). At any t, the value of the optimal hedge ratio gives the proportion of spot market positions held between t and t+1 that should be covered on the futures market.

To operationalize the time-varying optimal hedging rule, the covariance matrix of the prediction errors must be estimated at every t. This requires a model of how the variances and covariances evolve through time. Three such models are now considered.

III. Moving Sample Variances and Covariances

Suppose the covariance matrix of the prediction errors at t depends on magnitudes of the past n realized prediction errors according to

$$(7) \qquad H_t = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{t-i}\varepsilon'_{t-i}.$$

Then given (5) and (6), the covariance matrix at every t can be estimated by moving sample variances and covariances of the past n realized prediction errors (price changes). Once the moving sample variances and covariances have been computed, the optimal hedge ratio can be estimated using (3).

Equation (7) is very easy to implement since it contains no unknown parameters to be estimated. All one has to do is compute the past prediction errors using (5) and (6), choose n, and use the formula in (7). However, this approach is restrictive. It assumes that the weights on past squared and cross-multiplied prediction errors are constant through lags n and zero thereafter. The ARCH model provides a more flexible mechanism for accounting for heteroscedastic error processes.

IV. The ARCH Model

Consider the following generalization of (7):

$$(8) \qquad H_t = A'_0A_0 + \sum_{i=1}^{n} A'_i\varepsilon_{t-i}\varepsilon'_{t-i}A_i$$

where: A_0 is a (2x2) symmetric matrix of parameters; and
A_i are n (2x2) matrices of parameters for i=1,2,...,n.

Equation (8) is a multivariate positive definite parameterization of the ARCH(n) model introduced by Engle.[4] This model is clearly more general than the moving variances and covariances modeled in (7) because the weights on different lags are flexible and each equation contains a larger set of explanatory variables. However, the unknown parameters in (8) must now be estimated econometrically.

Maximum likelihood is the recommended procedure for estimating (8). Let

$$\Theta = \{\text{all parameters in } A_0, A_1, \ldots, A_n\}.$$

Assuming the prediction errors are normally distributed, then the conditional log likelihood function for a sample of T observations on the prediction errors is

$$(9) \qquad L(\Theta) = - T \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log|H_t(\Theta)| - \frac{1}{2} \sum_{t=1}^{T} \varepsilon'_t H_t^{-1}(\Theta)\varepsilon_t$$

where: $H_t(\Theta)$ is equation (8); and
ε_t is defined by (5) and (6).

Maximizing the log likelihood function with respect to Θ is a nonlinear optimization problem that must be solved with numerical methods. The algorithm used in this paper is based on Berndt, Hall, Hall and Hausman.

Equation (8) can also be estimated with OLS. Since the covariance matrix is symmetric, a trivariate vector autoregression is suggested

$$(10) \quad Vech(\varepsilon_t \varepsilon_t') = A_0^* + \sum_{i=1}^{n} A_i^* \, Vech(\varepsilon_{t-1} \varepsilon_{t-1}')$$

where: A_0^* is a (3x1) vector of parameters;
A_i^* are (3x3) matrices of parameters for i=1,2,...,n; and
Vech is the column stacking operator that stacks the lower triangular portion of a symmetric matrix.

Unrestricted estimation of (10) is straightforward using OLS on each equation. However, while OLS remains a consistent estimator for this case it is not efficient.

V. The GARCH Model

The GARCH model is a generalization of ARCH that was suggested by Bollerslev. Formally, consider the following GARCH (n,m) model for the conditional covariance matrix of the prediction errors:

$$(11) \quad H_t = A_0' A_0 + \sum_{i=1}^{n} A_i' \varepsilon_{t-i} \varepsilon_{t-1}' A_i + \sum_{j=1}^{m} B_j' H_{t-j} B_j$$

where; B_j are m (2x2) matrices of parameters for j=1,2,...,m.

Compared to the ARCH model, the additional terms involve lagged values of the conditional covariance matrix. Notice that there is an analogy between modeling conditional means with autoregressive moving average (ARMA) processes and modeling conditional variances and covariances with GARCH processes. In the GARCH model, terms involving past values of the conditional moments are like autoregressive components in ARMA models while terms involving past squared prediction errors are like moving average components. Hence, the GARCH model is a parsimonious parameterization of a wide class of heteroscedastic error processes in much the same way that ARMA models provide a parsimonious parameterization of a wide class of stationary stochastic processes.

Equation (11) can be estimated using maximum likelihood methods. The log likelihood function is still defined by (9) but the parameter vector, Θ, now contains

$$\Theta = \{\text{all parameters in } A_0, A_1, \ldots A_n \text{ and } B_1, B_2, \ldots, B_m\} .$$

Furthermore, (11) is now used as the model for the conditional covariance matrix. The maximization problem is highly nonlinear so numerical methods must be used. Again, the algorithm is based on Berndt, Hall, Hall and Hausman.

VI. An Application

The three approaches to estimating time-varying conditional covariance matrices were applied to weekly observations on wheat spot and futures prices. Having obtained the estimated conditional covariance matrix at each t, the conditional covariance between spot and futures and the conditional variance of futures were substituted into (3) to get a time path of optimal hedge ratios. The resulting hedge ratios are relevant for a storage hedge on wheat being held during the week from t to t+1.

Data are the mid-week (Wednesday) closing price on the relevant market. Spot prices are for the Saginaw market in Michigan and were obtained from Mid-States Terminals, Toledo, Ohio. Futures prices are for the May contract on the Chicago Board of Trade and were obtained from various issues of the Chicago Board of Trade Statistical Annual. The observation period runs from June 1977 to June 1985, a total of 410 observations.

As indicated earlier, a model must be chosen for the conditional means as well as for the conditional covariance matrix of spot and futures prices before optimal hedge ratios can be estimated. The model used here for the conditional means is simply that they equal the current realization of the respective prices. Thus, equations (5) and (6) are satisfied and the model for the conditional covariance matrix is estimated using price change data. Evidence supporting this simple model for the conditional means of weekly spot and futures prices for wheat can be found in Myers and Thompson.

There are no parameters to estimate in the simple moving sample variances and covariances model. After experimenting with different alternatives, a lag length of ten weeks was chosen for computing the moving sample variances and covariances. With the ARCH and GARCH models, however, parameters in the equation for the conditional covariance matrix had to be estimated econometrically. After some experimentation, ARCH(2) and GARCH(1,1) models were fitted to the data.[5]

Results from OLS estimation of the ARCH (2) model are shown in table 1. Parameter estimates, their t-values and the Durbin-Watson statistic are all provided. Also provided is the Lagrange multiplier statistic (T times the unadjusted coefficient of determination from the OLS regression) for testing the null hypothesis that the relevant conditional variance or covariance is constant (Engel). The probability values for the Lagrange multiplier test strongly support the existence of conditional heteroscedastic prediction errors.

Results from maximum likelihood estimation of the ARCH(2) and GARCH(1,1) models are shown in table 2. The chi-square statistics and corresponding probability values are from a likelihood ratio test of the hypothesis that the conditional covariance matrix of spot and futures prices in constant over time. The null is soundly rejected in both models. Of course, the likelihood ratio tests are based on the asymptotic distribution of the estimators.

How much influence does the time variation in the conditional covariance matrix of spot and futures prices have on the time path of optimal hedge ratios? To answer this question, optimal hedge ratios were calculated for every week of the sample period assuming: (a) the conditional covariance matrix is constant; (b) moving sample variances and covariances; (c) ARCH(2) estimated by OLS; (d) ARCH(2) estimated by maximum likelihood; and (e) GARCH(1,1) estimated by maximum likelihood. The time patterns of optimal hedge ratios over the sample period were then compared across models. Summary statistics that facilitate such a comparison are provided in table 3. The mean, standard deviation and extreme values of the weekly hedge ratios over the sample period were computed. The means are fairly consistent across models, indicating a hedge ratio of about 90%. However, each of the time-varying models indicate substantial fluctuations in the optimal hedge ratio - fluctuations that would be ignored if one took the conventional approach to optimal hedge ratio estimation. The method that provided the most volatile time pattern for optimal hedge ratios over the sample period was moving sample variances and covariances.

Given the fact that the GARCH model is the most general of those considered, and the very high t-values on two of the GARCH parameters, the GARCH(1,1) might be thought of as the best available model. A typical weekly deviation from the mean in the optimal hedge ratio under the GARCH(1,1) model is 11% and the high and low values for the hedge ratio over the sample period were 120% and 55% respectively. This indicates that substantial errors may be made if a constant optimal hedge ratio is recommended when prediction errors are actually conditionally heteroscedastic.

Since the GARCH model is the most difficult and costly to estimate, it is natural to ask whether the simpler models can provide a reasonable approximation to the GARCH hedge ratios. To answer this question the root mean squared deviation from the GARCH model was computed for each of the other models over the sample period. Results are shown in table 4. Perhaps surprisingly, the constant conditional covariance matrix model provides the minimum mean squared deviation away from the GARCH(1,1) hedge ratios, while the moving sample variances and covariances perform the worst. Provided the GARCH(1,1) is the right model, this indicates that (in this particular application) assuming the optimal hedge ratio to be constant may be a reasonable second best if you are not going to estimate the GARCH model. Nevertheless, there remains a substantial divergence between the constant hedge ratio estimate and the time pattern of hedge ratios from the GARCH model.

Finally, a comment on seasonality in optimal hedge ratios is in order. Various specifications of seasonal dummy variables were incorporated into the models for the conditional covariance matrix but none were found to be significant. This is surprising since one might expect price volatility to increase at times when the crop is most vulnerable to changing weather conditions. Nevertheless, statistical tests and an estimation of optimal hedge ratio plots over a number of years provided no evidence of seasonality in optimal hedge ratios estimated with the GARCH(1,1) model.[6] Optimal hedge ratio plots for 1983/84 and 1984/85 are shown in figure 1.

VII. Conclusions

Optimal hedge ratios are not constant over time - as the volatilities of spot and futures prices fluctuate, so the optimal hedge ratio changes. It is therefore important to model the heteroscedasticity in the conditional covariance matrix of spot and futures prices, thereby estimating a time pattern of optimal hedge ratios. Moving sample variances and covariances are one method for doing this but ARCH and GARCH models provide a much more flexible and general approach. The difficulty with the latter models is that they must be estimated using nonlinear maximum likelihood methods.

An application to storage hedging of wheat in Michigan indicated substantial errors would be made by assuming optimal hedge ratios are constant over time. There was strong evidence of heteroscedasticity in the conditional covariance matrix of weekly spot and futures data for wheat prices, and the GARCH(1,1) model seemed the most appropriate of alternatives considered. Substantial weekly changes in optimal hedge ratios estimated with the GARCH(1,1) model were common with a typical deviation from the mean having an order of magnitude of 11 percentage points. Thus, the typical practice of assuming a constant optimal hedging rule may lead to errors in the recommended hedge ratio.

ENDNOTES

1 This optimal hedging rule relies on the assumptions that futures markets are unbiased and that output is nonstochastic, but little need be assumed about the hedger's utility function (Benninga, Eldor and Zilcha).

2 The estimated hedge ratio may change if the data set is updated as more data becomes available. However, this just reflects a reduction in the sample variance of the estimator. The underlying population variances and covariances generally are assumed fixed.

3 A futures market is unbiased if the futures price expected to hold at t+1 (given information available at t) is equal to the realized futures price at time t.

4 The positive definite parameterization restricts the conditional covariance matrix to be symmetric and positive definite at every t.

5 The experimentation involved looking at the autocorrelations of squared price changes and fitting a range of alternative models. The final models chosen gave the most consistent and sensible results.

6 This is consistent with the weak seasonality found by Fackler in a univariate GARCH model of corn prices.

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Table 1

OLS Estimation of the ARCH(2) Model

Explanatory Variables	Dependent Variable: u_t^2	$u_t v_t$	v_t^2
	Parameter Estimate (t-statistic)		
Constant	0.011 (6.518)	0.009 (6.265)	0.010 (6.690)
u_{t-1}^2	0.279 (2.371)	0.109 (1.071)	0.115 (1.071)
$u_{t-1}v_{t-1}$	-0.244 (-0.841)	-0.094 (-0.375)	-0.273 (-1.037)
v_{t-1}^2	0.219 (1.076)	0.187 (1.066)	0.335 (1.814)
u_{t-2}^2	-0.103 (-0.879)	-0.038 (-0.375)	-0.053 (-0.501)
$u_{t-2}v_{t-2}$	0.349 (1.205)	0.167 (0.666)	0.216 (0.819)
v_{t-2}^2	-0.211 (-1.040)	-0.146 (-0.833)	-0.169 (-0.915)
Durbin-Watson	2.00	1.99	1.99
TR^2	26.96	17.93	14.74
Probability Value	0.000	0.006	0.022

Table 2

Maximum Likelihood Estimation of the ARCH and GARCH Models

	ARCH(2)	GARCH(1,1)
Parameter	Parameter Estimate (t-statistic)	
$A_0(1,1)$	0.060 (0.206)	0.026 (1.875)
$A_0(2,1)$	0.090 (0.456)	0.009 (0.223)
$A_0(2,2)$	0.053 (0.181)	0.018 (1.394)
$A_1(1,1)$	0.222 (1.837)	0.153 (1.530)
$A_1(1,2)$	-0.146 (-1.126)	-0.112 (-1.098)
$A_1(2,1)$	0.115 (1.04)	0.219 (2.285)
$A_1(2,2)$	0.451 (3.439)	0.369 (3.998)
$A_2(1,1)$	-0.159 (-0.829)	
$A_2(1,2)$	-0.035 (-0.169)	
$A_2(2,1)$	-0.201 (-1.002)	
$A_2(2,2)$	0.057 (0.261)	
$B_1(1,1)$		0.922 (16.248)
$B_1(1,2)$		0.021 (0.397)
$B_1(2,1)$		-0.012 (-0.200)
$B_1(2,2)$		0.926 (17.520)
Log Likelihood	1265.46	1266.01
χ^2 (8)	61.78	62.88
Probability Value	0.000	0.000

Table 3

Summary Statistics for Time-Varying Optimal Hedge Ratios Between June 1977 and June 1985

Model	Mean	Standard Deviation	High	Low
Constant Conditional Covariance Matrix	0.91	0	0.91	0.91
Moving Sample Variances and Covariances	0.92	0.23	1.73	0.26
ARCH(2) Estimated by OLS	0.91	0.08	1.32	0.56
ARCH(2) Estimated by Maximum Likelihood	0.92	0.07	1.12	0.46
GARCH(1,1)	0.89	0.11	1.20	0.55

Table 4

Root Mean Squared Deviations From the GARCH(1,1) Model
Between June 1977 and June 1985

Model	Root Mean Squared Deviation
Constant Conditional Covariance Matrix	0.11
Moving Sample Variances and Covariances	0.20
ARCH(2) Estimated by OLS	0.12
ARCH(2) Estimated by Maximum Likelihood	0.12

FIGURE 1

Optimal Hedge Ratios for 1983/84 and 1984/85