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CONSTANT OR TIME-VARYING OPTIMAL HEDGE RATIOS?

Giancarlo Moschini and Satheesh Aradhyula¹

1. Introduction

Individuals or firms with risky cash positions can eliminate part or all of their risk by hedging using futures contracts. In this setting, the analysis of the optimal hedging problem typically makes use of the concept of a 'hedge ratio' -- the amount of futures bought or sold expressed as a proportion of the cash position. Under certain restrictive conditions that apply, nonetheless, to a number of practical situations, the optimal hedge ratio is given by the ratio of the covariance of futures and cash prices to the variance of futures prices. Essentially, this result applies to situations with price risk only (no quantity uncertainty), and reflects the effects of (unhedgeable) basis risk. Under these conditions an hedge ratio equal to the ratio of covariance between cash and futures prices to the variance of futures prices is always optimal in the sense of minimizing the variance of profit of a (given) cash position (Kahl, 1983). Furthermore, if the futures price is unbiased, then such a hedge ratio is also optimal in the sense of maximizing the expected utility of profit given certain further restrictions of the distribution of cash and futures prices (Benninga, Eldor, and Zilcha, 1984; Myers, 1991).

The implementation of this simple hedging rule has traditionally relied on estimating the optimal hedge ratio by the slope of a ordinary least square regression of cash on futures prices, but more general procedures been proposed recently. First, there is a need to choose an estimating model such that the variance and covariance of the optimal hedge ratio are defined relative to the proper conditional mean (Myers and Thompson, 1989). For example, if cash and futures price follow a martingale process, then the slope of a regression of cash price changes on futures price changes (and not the slope of a regression in levels) estimates the relevant ratio. Moreover, it has been recognized that commodity prices, along with many other financial series, display time-varying volatility. If the variance and covariance of cash and futures prices vary over time, then the optimal hedge ratio may vary over time, which calls for estimation techniques consistent with such an hypothesis.

To allow for time-varying optimal hedge ratios Checchetti, Cumby, and Figlewski (1988) model cash and futures prices with Engle's ARCH framework, while Baillie and Myers (1991), and Myers (1991) use Bollerslev's GARCH approach. Although the results of these studies point to the existence of considerable time variation in the volatility of commodity prices, consistent with recent evidence concerning other financial time series, the argument in favor of time-varying hedge ratios needs further analysis because it is not strictly implied by time varying volatility. In other words, it is quite possible to have a time varying distribution of futures and cash prices where the ratio of covariance to variance of futures is time invariant, a condition that seems to have been neglected in

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the studies mentioned above. Hence, the first objective of this paper is to develop a framework that allows specification and testing of time dependent distributions of cash and futures prices that yield a constant hedge ratio.

An additional reason to re-examine the issue of time-varying optimal hedges, even if the time varying distribution of cash and futures price entails a time-varying hedge ratio, is the fact that in most cases the actual variability of this ratio is sensitive to the time horizon of the futures hedge. As the time at which the hedge is lifted moves into the future, the relevant conditional moments converge (under stationarity) to the time-invariant unconditional moments, such that the relevant hedge ratio may yet be fairly constant. The second objective of this paper is to discuss this issue in some detail.

2. The Hedging Model

Following Myers (1991), assume that at some point in time a risk-averse individual allocates an initial wealth among a risk-free asset and a risky asset (an asset whose end-of-period return is random). There is a futures market for the risky asset, but the hedging opportunity are imperfect because of basis risk. Hence the problem of the individual could be represented as:

$$\max_{q_{t-1}, x_{t-1}} E[U(W_t) | \Omega_{t-1}] \quad (1)$$

where:

$$W_t = (1+r)[W_{t-1} - p_{t-1}q_{t-1}] + p_t q_{t-1} + (f_{t-1} - f_t)x_{t-1} \quad (2)$$

The time subscripts $(t-1)$ and t denote end-of-period and beginning-of-period, respectively, W is wealth, q is the quantity of risky assets, x is the quantity sold in the futures market, p is the price of the risky asset, f is the futures price, r is the interest rate, E is the expectation operator, Ω is the information set. The random variables in (2) are p_t and f_t , the cash and futures prices at the end of the period.

Myers (1991) shows that, if (p_t, f_t) are jointly normally distributed, and the futures market is unbiased in that $f_{t-1} = E[f_t | \Omega_{t-1}]$, then the expected utility maximizing level of futures sales relative to the optimal level of investment on the risky asset (the optimal hedge ratio denoted HR_{t-1}) is:

$$\frac{x_{t-1}^*}{q_{t-1}^*} = \frac{\text{Cov}(p_t, f_t | \Omega_{t-1})}{\text{Var}(f_t | \Omega_{t-1})} = HR_{t-1} \quad (3)$$

a result similar to that of Benninga, Eldor, and Zilcha (1984). Alternatively, the hedge ratio HR_{t-1} can be viewed as a risk-minimizing optimal hedge given a level q_{t-1} of the risky asset (Kahl, 1983). If the joint distribution of cash and futures prices changes over time, then $\text{Var}(f_t | \Omega_{t-1})$ and $\text{Cov}(p_t, f_t | \Omega_{t-1})$ will typically change over time, such that the expected-utility maximizing hedge ratio in (3) will also change over time (Checchetti, Cumby, and Figlewski, 1988;

Baillie and Myers, 1991; Myers, 1991). The time path of this hedge ratio can be calculated given knowledge of the (time dependent) process generating cash and futures prices, which can be estimated with GARCH-type models.

It should be observed, however, that the hedge ratio in (3) can still be constant, even if $\text{Var}(f_t|\Omega_{t-1})$ and $\text{Cov}(p_t, f_t|\Omega_{t-1})$ both vary over time, as long as the covariance term is proportional to the variance term, i.e., $\text{Cov}(p_t, f_t|\Omega_{t-1}) = \gamma \text{Var}(f_t|\Omega_{t-1})$ for all t (for some constant γ). This case is somewhat restrictive, but perhaps not too unreasonable. To illustrate, it is useful to make explicit the link between cash and futures prices, the basis relationship, because the crux of the matter here is basis risk.² In particular, assume that cash and futures price are generated by the following stochastic process:

$$f_t = E[f_t|\Omega_{t-1}] + a_1 \epsilon_t + a_2 \eta_{1t} \quad (4)$$

$$p_t = E[p_t|\Omega_{t-1}] + b_1 \epsilon_t + b_2 \eta_{2t} \quad (5)$$

where $(\epsilon, \eta_1, \eta_2)$ are conditionally independently distributed, mean-zero random terms. Hence, cash and futures prices depend on a common shock (ϵ), but also on specific own disturbances (η_1, η_2). From (4) and (5):

$$\text{Cov}(p_t, f_t|\Omega_{t-1}) = a_1 b_1 \text{Var}(\epsilon_t|\Omega_{t-1}) \quad (6)$$

$$\text{Var}(f_t|\Omega_{t-1}) = a_1^2 \text{Var}(\epsilon_t|\Omega_{t-1}) + a_2^2 \text{Var}(\eta_{1t}|\Omega_{t-1}) \quad (7)$$

Hence, in this case $\text{Cov}(p_t, f_t|\Omega_{t-1})$ is not proportional to $\text{Var}(f_t|\Omega_{t-1})$, and the optimal hedge ratio is in fact time varying. Note, however, that this requires that the futures price may change without affecting the cash price (via the term η_1), which is indeed the necessary condition in terms of basis relationships for the optimal hedge ratio to be time varying. If movements in the futures price are always reflected in the cash price, however, $a_2 = 0$ and $\text{Cov}(f_t, p_t|\Omega_{t-1}) = (a_1/b_1) \text{Var}(f_t|\Omega_{t-1})$, which implies that the optimal hedge ratio is constant and equal to (a_1/b_1) . In conclusion, although a constant hedge ratio is restrictive, it is a legitimate possibility, even in the face of time-varying price distributions. Because of its obvious practical importance, it may be of interest to test this hypothesis. The next section shows how to implement bivariate GARCH models that allows for such test.

3. GARCH Models with Constant Optimal Hedge Ratios

Modeling the time dependence of cash and futures price distributions begins with an explicit model for the conditional mean of these prices. Following Baillie and Myers (1991), and Myers (1991), we will assume the following model of price formation:

² Without basis risk the optimal hedge ratio is obviously constant for all time periods, regardless of the possible time dependence of the distribution of the futures price.

$$p_t = p_{t-1} + \mu_t Z_t + u_{1t} \quad (8)$$

$$f_t = f_{t-1} + u_{2t} \quad (9)$$

Hence, consistent with the assumption leading to the optimal hedge ratio in (3) the futures price is assumed an unbiased predictor of the next period futures price ($E[f_t | \Omega_{t-1}] = f_{t-1}$), whereas for the cash market price $E[p_t | \Omega_{t-1}] = p_{t-1} + \mu_{1t} Z_t$ where the drift term $\mu_{1t} Z_t$ may denote, among other things, the effects of carrying charges and convenience yield. Equations (8) and (9) can be expressed in vector notation as:

$$\Delta y_t = \mu Z_t + u_t \quad (10)$$

where $\Delta y_t = (p_t - p_{t-1}, f_t - f_{t-1})'$, $\mu = (\mu_1, 0)$, and $u_t = (u_{1t}, u_{2t})'$, with the superscripted prime indicating vector transposition.

3.1 Positive Definite Multivariate GARCH(1,1)

Alternative parameterizations of the time dependence of the conditional covariance matrix $H_t = E[u_t u_t' | \Omega_{t-1}]$ are possible. Baillie and Myers (1991), and Myers (1991), consider two versions of the bivariate GARCH model of Engle. The GARCH(1,1) referred to as 'positive definite' parameterization in matrix notation is written as:

$$H_t = C' C + A' u_{t-1} u_{t-1}' A + B' H_{t-1} B \quad (11)$$

where C , A , and B are 2×2 parameter matrices (A and B are unrestricted and C is symmetric). If h_t^{ij} denote the elements of the conditional covariance matrix H_t , such that $h_t^{11} = \text{Var}(p_t | \Omega_{t-1})$, $h_t^{22} = \text{Var}(f_t | \Omega_{t-1})$, and $h_t^{12} = \text{Cov}(p_t, f_t | \Omega_{t-1})$, then equation (11) for the special case of a bivariate problem can be rewritten as:

$$h_t^{11} = \omega_{11} + a_{11}^2 u_{1t-1}^2 + 2a_{11}a_{21}u_{1t-1}u_{2t-1} + a_{21}^2 u_{2t-1}^2 + b_{11}^2 h_{t-1}^{11} + 2b_{11}b_{21}h_{t-1}^{12} + b_{21}^2 h_{t-1}^{22} \quad (12)$$

$$h_t^{12} = \omega_{12} + a_{11}a_{12}u_{1t-1}^2 + (a_{12}a_{21} + a_{11}a_{22})u_{1t-1}u_{2t-1} + a_{21}a_{22}u_{2t-1}^2 + b_{11}b_{12}h_{t-1}^{11} + (b_{12}b_{21} + b_{11}b_{22})h_{t-1}^{12} + b_{21}b_{22}h_{t-1}^{22} \quad (13)$$

$$h_t^{22} = \omega_{22} + a_{12}^2 u_{1t-1}^2 + 2a_{12}a_{22}u_{1t-1}u_{2t-1} + a_{22}^2 u_{2t-1}^2 + b_{12}^2 h_{t-1}^{11} + 2b_{12}b_{22}h_{t-1}^{12} + b_{22}^2 h_{t-1}^{22} \quad (14)$$

where a_{ij} are the elements of the A matrix, b_{ij} are the elements of the B matrix, and $\omega_{11} = (c_{11}^2 + c_{12}^2)$, $\omega_{22} = (c_{12}^2 + c_{22}^2)$, and $\omega_{12} = (c_{11} c_{12} + c_{12} c_{22})$.

Because in this notation the optimal hedge ratio is $HR_t = h_t^{12}/h_t^{22}$, this ratio will be constant if $h_t^{12} = \gamma h_t^{22}$ for all t , for some proportionality constant γ . For this to happen, the coefficients in (13) must be proportional to the corresponding coefficients in (14), which requires the following 5 nonredundant

restrictions:³

$$\begin{aligned} c_{12} &= \gamma c_{22}, & c_{11} &= \gamma^2 c_{22} \\ a_{11} &= \gamma a_{12}, & a_{21} &= \gamma a_{22} \\ b_{11} &= \gamma b_{12}, & b_{21} &= \gamma b_{22} \end{aligned} \quad (15)$$

Hence, the 11 independent parameters of the GARCH model (12)-(14) reduce to 6 with the restrictions in (15), yielding a restricted GARCH model with time-varying variances and covariance but with a constant hedge ratio.

How useful is this restricted GARCH parameterization? To answer this question, note that the restrictions in (15) when applied to (12) imply $h_t^{11} = \gamma^2 h_t^{22}$. Hence, the conditional correlation between cash and futures prices, $\rho_t = h_t^{12} / (h_t^{11} h_t^{22})^{1/2}$, is restricted to equal unity. This appears very restrictive, because it corresponds to the case of no basis risk. In terms of the linear basis relationship postulated in (4)-(5), a correlation coefficient equal to one would require $a_2 = b_2 = 0$. Hence, for the purpose of obtaining a constant optimal hedge ratio, the restricted GARCH model of equations (12)-(15) is clearly unnecessarily too restrictive, because in the context of (4)-(5) all that one needs is $a_2 = 0$, as discussed earlier.

3.2 A More General Multivariate GARCH(1,1)

A GARCH parameterization alternative to (12)-(14), also considered by Myers (1991) and Baillie and Myers (1991), is:

$$\text{vech}(H_t) = C + A \text{vech}(u_{t-1} u_{t-1}') + B \text{vech}(H_{t-1}) \quad (16)$$

where vech is the column stacking operator that stacks the lower triangular portion of a symmetric matrix, and for the bivariate case C is a 3x1 vector of parameters, and A and B are 3x3 matrices of parameters. Hence, this model has a total of 21 parameters, and for the bivariate case can be written explicitly as:

$$h_t^{11} = \omega_1 + \alpha_{11} u_{1t-1}^2 + \alpha_{12} u_{2t-1}^2 + \alpha_{13} u_{1t-1} u_{2t-1} + \beta_{11} h_{t-1}^{11} + \beta_{12} h_{t-1}^{22} + \beta_{13} h_{t-1}^{12} \quad (17)$$

$$h_t^{22} = \omega_2 + \alpha_{21} u_{1t-1}^2 + \alpha_{22} u_{2t-1}^2 + \alpha_{23} u_{1t-1} u_{2t-1} + \beta_{21} h_{t-1}^{11} + \beta_{22} h_{t-1}^{22} + \beta_{23} h_{t-1}^{12} \quad (18)$$

$$h_t^{12} = \omega_3 + \alpha_{31} u_{1t-1}^2 + \alpha_{32} u_{2t-1}^2 + \alpha_{33} u_{1t-1} u_{2t-1} + \beta_{31} h_{t-1}^{11} + \beta_{32} h_{t-1}^{22} + \beta_{33} h_{t-1}^{12} \quad (19)$$

where α_{ij} , β_{ij} , and ω_i are the elements of the matrices A , B , and C , respectively.

A restricted GARCH model with constant optimal hedge ratios can be obtained readily from equations (17)-(19) by requiring $h_t^{12} = \gamma h_t^{22}$, which yields the

³ The 6 equalities in (15) yield 5 restrictions because γ needs to be estimated.

following 6 nonredundant restrictions (again, γ needs to be estimated):

$$\begin{aligned}\alpha_{31} &= \gamma\alpha_{21}, & \alpha_{32} &= \gamma\alpha_{32} \\ \alpha_{33} &= \gamma\alpha_{23}, & \beta_{31} &= \gamma\beta_{21} \\ \beta_{32} &= \gamma\beta_{22}, & \beta_{33} &= \gamma\beta_{23} \\ \omega_3 &= \gamma\omega_2.\end{aligned}\tag{20}$$

This restricted GARCH model appears more useful for the purpose of testing the constancy of optimal hedge ratios because it does not unnecessarily restrict h_t^{11} , and therefore ρ_t . However, the problem with such model is that the positive-definiteness restrictions on the conditional covariance matrix is not maintained by parametric restrictions.

3.3 Special Cases of Multivariate GARCH(1,1)

The bivariate GARCH(1,1) in (17)-(19) is often estimated in a more parsimonious version obtained by assuming that A and B are diagonal matrices, as originally suggested by Bollerslev, Engle and Wooldridge (1988), and used by Baillie and Myers (1991) and Myers (1991) to estimate time-varying optimal hedge ratios. The resulting GARCH model is:

$$h_t^{11} = \omega_{11} + \alpha_{11}u_{1t-1}^2 + \beta_{11}h_{t-1}^{11}\tag{21}$$

$$h_t^{22} = \omega_2 + \alpha_{22}u_{2t-1}^2 + \beta_{22}h_{t-1}^{22}\tag{22}$$

$$h_t^{12} = \omega_3 + \alpha_{33}u_{1t-1}u_{2t-1} + \beta_{33}h_{t-1}^{12}\tag{23}$$

Hence, this formulation can yield a constant hedge ratio h_t^{12}/h_t^{22} only if $\alpha_{11} = \beta_{11} = \alpha_{22} = \beta_{22} = \alpha_{33} = \beta_{33} = 0$, which would imply that the variances of futures and cash prices and their covariance are constant over time (i.e., no GARCH effect at all). Clearly, this is extremely restrictive, and does not provide a useful vehicle to test the constancy of hedge ratios over time.

Another parsimonious version of the bivariate GARCH(1,1) model, which has computational advantages, assumes that the conditional correlation coefficient is constant (Bollerslev, 1990). In other words, the GARCH(1,1) model is written as:

$$h_t^{11} = \omega_1 + \alpha_{11}u_{1t-1}^2 + \beta_{11}h_{t-1}^{11}\tag{24}$$

$$h_t^{22} = \omega_2 + \alpha_{22}u_{2t-1}^2 + \beta_{22}h_{t-1}^{22}\tag{25}$$

$$h_t^{12} = \rho \sqrt{h_t^{11}} \sqrt{h_t^{22}}\tag{26}$$

where ρ is the (constant) conditional correlation between cash and futures prices. However, note that the conditional correlation and the optimal hedge ratio are related as:

$$HR_t = \rho_t \sqrt{\frac{h_t^{11}}{h_t^{22}}} \quad (27)$$

Hence, ρ_t and HR_t can both be constant only if the ratio h_t^{11}/h_t^{22} is constant. This is very restrictive, more than is necessary to obtain a constant conditional hedge ratio. For example, in terms of the basis relationship (4)-(5) one would require $a_2 = b_2 = 0$, again implying that the correlation is not only constant but also equal to unity (i.e., there is no basis risk). In other words, if one starts with a GARCH specification that assumes constant conditional correlation, then it is very unlikely that the hedge ratio will be constant. For example, even when the true HR_t is constant, if the conditional correlation is not constant specifying a GARCH with constant ρ is likely to yield time varying HR_t .

The analysis of the constant conditional correlation model, however, suggests another useful parameterization of the bivariate GARCH model that allows a constant hedge ratio specification. Specifically, the 'diagonal' GARCH model of equations (21)-(23) can be augmented by rewriting the covariance equation (23) as:

$$h_t^{12} = \omega_3 + \alpha_{33}u_{1t-1}u_{2t-1} + \beta_{33}h_{t-1}^{12} + \gamma h_{t-1}^{22} \quad (28)$$

The bivariate GARCH model of equations (21)-(22) and (28) would display a constant hedge ratio if the following three restrictions are satisfied:

$$\omega_3 = \alpha_{33} = \beta_{33} = 0 \quad (29)$$

a proposition that can be subjected to statistical test.

4. Distant Conditional Hedges

The notion of time-varying optimal hedge ratio as addressed in this paper, and in the studies of Checchetti, Cumby, and Figlewski (1988), Baillie and Myers (1991), and Myers (1991), essentially deals with an estimation problem because the problem in (1) and (2) is not dynamic in a structural sense.⁴ In particular, (1) and (2) assume that a cash and futures position is entered at time (t-1) and is liquidated one period hence [in Myers (1991), and in our subsequent empirical application, this period is one week]. For most relevant hedging cases, however, the cash position is held for a longer time. For instance, given that we are considering only basis risk and are abstracting from production risk, our hedging situation best applies to storage hedges. Storage is likely to be undertaken for periods much longer than one week. If adjusting the hedge position were costless, then the time-varying optimal hedge would be

⁴ Hence, this sort of time-varying hedge ratios should be carefully distinguished from the case when the revision of the hedge ratio is brought about by the gradual resolution of uncertainty, as in Anderson and Danthine (1983), Karp (1988), and others.

an optimal strategy for such a case. However, the existence of transactions costs may make such continuous updating of the hedge position unattractive. In such a case the agent may wish to enter the futures position only once (when the cash position is opened), and carry it as long as the risky cash position is held.

If conditional moments change over time, the relevant hedge ratio is still time dependent even if there is no plan to adjust the hedge position over time. Specifically, suppose an hedge is entered at time $(t-1)$ to protect a cash position maturing not at time t but at time $(t+k)$. Assuming that the futures position to be established will be liquidated only when the cash position is liquidated, the relevant conditional hedge ratio is:

$$HR_{t+k} = \frac{Cov(p_{t+k}, f_{t+k} | \Omega_{t-1})}{Var(f_{t+k} | \Omega_{t-1})} \quad (30)$$

To make this rule operational, note that the conditional variances and covariance of a GARCH model are random variables. Thus, they are not known more than one period ahead, and the estimated optimal conditional hedge ratio in (28) is:

$$HR_{t+k} = \frac{E[h_{t+k}^{12} | \Omega_{t-1}]}{E[h_{t+k}^{22} | \Omega_{t-1}]} \quad (31)$$

If the optimal hedge ratio in (29) is estimated from a (diagonal) GARCH(1,1) model, as in (24)-(26), following Baillie and Bollerslev (1992) we have:

$$E[h_{t+k}^{22} | \Omega_{t-1}] = \omega_2 \sum_{j=0}^{k-1} (\alpha_{22} + \beta_{22})^j + (\alpha_{22} + \beta_{22})^k h_t^{22} \quad (32)$$

$$E[h_{t+k}^{12} | \Omega_{t-1}] = \omega_3 \sum_{j=0}^{k-1} (\alpha_{12} + \beta_{12})^j + (\alpha_{12} + \beta_{12})^k h_t^{12} \quad (33)$$

If the GARCH(1,1) is stationary, such that $(\alpha_{22} + \beta_{22}) < 1$ and $|\alpha_{12} + \beta_{12}| < 1$ and the unconditional moments are defined, the above expression is equivalent to:

$$E[h_{t+k}^{22} | \Omega_{t-1}] = \sigma_{22}^2 + (\alpha_{22} + \beta_{22})^k (h_t^{22} - \sigma_{22}^2) \quad (34)$$

$$E[h_{t+k}^{12} | \Omega_{t-1}] = \sigma_{12}^2 + (\alpha_{12} + \beta_{12})^k (h_t^{12} - \sigma_{12}^2) \quad (35)$$

where σ_{22}^2 is the unconditional variance futures price and σ_{12}^2 is the unconditional covariance of futures and cash prices.

As k increases (the hedge will be lifted further and further ahead in the future) the quantity $(\alpha_{rs} + \beta_{rs})^k$ approaches zero and the conditional hedge ratio approaches the unconditional hedge ratio $\sigma_{12}^2 / \sigma_{22}^2$. Hence, the hedge ratio in (8) is going to look fairly constant as k increases. Note, however, that this is

somewhat distinct from the issue of statistical significance. As long as $(\alpha_{rs} + \beta_{rs})$ is statistically different from zero, $(\alpha_{rs} + \beta_{rs})^k$ will be statistically different from zero, and the conditional moments will differ statistically from the unconditional ones no matter how small $(\alpha_{rs} + \beta_{rs})^k$ gets (unless the proportionality $h_t^{12} = \gamma h_t^{22}$ holds for all t). This is a good reminder, perhaps, of the distinction between statistical and economic significance.

The existence of the unconditional moments in the GARCH model, and therefore of the unconditional hedge ratio, depends on the condition of stationarity, which may or may not hold. If it does hold, and assuming that the unconditional hedge ratio is what one may want (because k is 'large enough', say), then a relevant issue may be what is best suited to get the unconditional estimates, say OLS or GARCH (assuming that the true process is, in fact, conditionally heteroscedastic).

5. An Application to Iowa Corn

The model outlined in the foregoing is applied to the problem of hedging cash corn in Iowa. Our framework is very similar to that of Myers (1991). Essentially, it is assumed that an investor (say, an elevator operator) buys and stores corn for resale at a price which is unknown at the time of purchase. The investor can hedge the long cash position by selling futures. We assume that the reference contract is the July contract (when a contract approaches maturity, at the end of June, the July contract of the following year is used.) The investor takes out futures positions on this contract and reevaluates his portfolio on a weekly basis. The portfolio may be adjusted every week to reflect changing information.

The price series used are mid-week (Thursday) prices. The cash price is the average of corn cash prices quoted in North-Central Iowa. The futures price is the Thursday closing price for the July Corn contract quoted on the Chicago Board of Trade. The sample period extends from January 1976 through December 1990, with a total of 782 observations.

Two different specifications of bivariate GARCH models were estimated using maximum likelihood methods. The variance-covariance matrix (H_t) in full vech specification has 21 parameters and is given by equations (17)-(19). Constancy of the OHR in this case is obtained if the restrictions in (20) are satisfied. The alternative specification we considered is the augmented diagonal specification of equations (21), (22) and (28). The augmented diagonal formulation can yield a constant OHR if the restrictions in (29) hold. Note that even under (29), the variance-covariance matrix is time-varying. The augmented diagonal specification has only 10 parameters in H_t and is more parsimonious than the full vech specification. GARCH models were estimated separately for corn and soybeans.

Estimation of OHRs requires specifying a model for the conditional means of cash and futures prices. The specification that we choose lead to the following model of cash and futures prices:

$$\Delta p_t = \delta_0 + \delta_1 DUM_t + u_{1t} \quad (36)$$

$$\Delta f_t = u_{2t}$$

(37)

Hence, we assume that the expected return to holding futures is zero, so that the conditional mean of the futures price can be specified as $E[f_t|\Omega_{t-1}] = f_{t-1}$. Despite its simplicity, there is considerable evidence supporting this Martingale process for commodity futures price data (Gordon, 1985). For the conditional mean of cash prices we are assuming $E[p_t|\Omega_{t-1}] = p_{t-1} + \delta_0 + \delta_1 \text{DUM}_t$. As in Myers (1991), the constant δ_0 captures the fact that (other things being equal) the cash price rises throughout the crop year to reflect carrying charges and convenience yield. However, because we are using data spanning many crop years, we have to allow for the fact that (other things being equal) price drop just before harvest every year. This effect is captured by the term involving the DUM_t , a dummy variable taking value of one for weeks in the months of July, August and September, and value zero otherwise.

6. Results

Table 1 reports the maximum likelihood estimates of the conditional mean of cash prices for corn and soybeans assuming that the variance-covariance matrix is constant (time-invariant). All parameters in Table 1 are significant at all conventional levels of significance. The Ljung-Box statistics for the residuals are slightly below 51.0, the relevant critical value at the 5% level. Hence, the null hypothesis that the residuals from each estimated conditional mean equation are white noise is not rejected, implying that the simple conditional models of cash and futures prices seem to fit the data adequately. When squared residuals are examined, however, we find serial correlation among them. In all cases, the Q^2 statistic at 36 degrees of freedom is significant at 1% level. This correlation among squared residuals suggests that a GARCH model may be appropriate (Bollerslev, 1987).

Maximum likelihood estimates for the corn hedging model under full vech specification are presented in Table 2. Results in Table 2 indicates that 17 out of 23 model parameters for corn hedging model are significant at 5% level. As expected, δ_0 is positive and δ_1 is negative. The Ljung-Box statistics for the standardized residuals, and the standardized squared residuals, from the estimated GARCH models are also reported in Table 2. In each case, the estimated values for $Q(36)$ and $Q^2(36)$ are below 51.0, the critical value of the chi-squared distribution at the 5% level. Thus, no further first- or second-order serial dependence is present in the estimated GARCH models.

To test for constancy of OHRs, the full vech GARCH model is re-estimated after imposing restrictions in (20). Because the restricted model under (20) is nested in the full model, standard likelihood ratio tests can be used for testing the constancy of the OHRs. The appropriate statistic is twice the difference of the maximized values of the log likelihood functions for the unconstrained and constrained models which will have a chi-square distribution with 6 degrees of freedom under the null hypothesis. The calculated statistic for corn (72.52) is well above the critical value of the chi-square statistic with six degrees of freedom at all conventional levels of significance. Thus, for Iowa corn the null hypothesis of constant OHRs can be convincingly rejected in favor of time-varying OHRs. It should be noted that our test is somewhat more general than previous

ones (e.g., Myers 1991, Baillie and Myers 1991) in that the variance-covariance matrix, even under the null hypothesis of constant OHR, is time-varying.

The estimated hedging models for corn under the augmented diagonal specification is reported in Table 3. The GARCH parameter estimates in Table 3 are highly significant. Again, the Q and Q^2 test statistics reported in Table 3 are statistically insignificant for the unrestricted GARCH model. However, the restricted model displays considerable correlation of the squared residuals in the futures price equation. As in the full vech specification, a likelihood ratio test can be used to test formally for constancy of OHRs. The calculated test statistic for corn (220.28) is well above the critical value of the chi-square distribution with 3 degrees of freedom. Hence, the null hypothesis of constant OHRs can be rejected under augmented diagonal specification as well. Thus, results in tables 3 and 4 indicate that OHRs are indeed time-varying.

OHRs are computed from the estimated GARCH models using the in sample estimates of the time-varying conditional variance-covariance matrices. Estimated conditional OHR's under full vech specification are illustrated in Figure 1. The horizontal straight line is the unconditional OHR implied by the unconditional moments of the H_t , also under full vech specification. Figure 1 clearly demonstrates that OHRs are time-varying and change from week to week as new information is obtained. The conditional OHRs presented in Figures 1 and 2 are obtained by using all the information available prior to the present period. Following the discussion of section 4, k-period ahead OHRs are also computed using the equation (29). Figure 2 illustrates six-weeks ahead OHRs for corn under the full vech specification. It is apparent from these figures that as we forecast farther and farther into the future, the conditional OHRs converge very quickly to the unconditional OHRs.

7. Conclusion

In this paper we have developed a general model for testing the hypothesis that the optimal hedge ratio is constant over time. In particular, we have developed alternative parameterizations of the bivariate GARCH(1,1) model that nest the hypothesis of constancy of the ratio of conditional covariance to conditional variance of one of the variables, and that retain time-varying variances and covariances even under the restrictive assumption of constant OHR.

Two of the GARCH models developed were estimated using weekly data for Iowa corn. In both cases the statistical test suggests that the OHR is indeed time varying. However, when longer holding periods are considered, the time-varying OHR converges fairly rapidly to the constant hedge ratio based on the unconditional moments. Hence, for some realistic hedging problems a constant hedge ratio may be desirable, notwithstanding the fact that statistically the time-varying OHR dominates the constant ones. What remains to be seen is whether the time-varying hedge ratio performs significantly better than a constant one from an economic standpoint. As Myers (1991) has found in a similar context, the difference in performance of time-varying and constant hedge ratios may be small.

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Table 1. Maximum Likelihood Estimates of Constant Covariance Matrix Hedging Model for Corn

Parameter	Estimate	t-value
δ_0	0.00069	3.636
δ_1	-0.00218	-4.554
h_{11}	0.00606	9.672
h_{22}	0.00501	19.662
h_{12}	0.00446	17.605
Q(36)	Δp_t	50.50
Q(36)	Δf_t	49.11
$Q^2(36)$	Δp_t	378.33
$Q^2(36)$	Δf_t	382.59
Log-likelihood		2260.27

Notes:

Q(36), and $Q^2(36)$ denote the Ljung-Box Portmanteau test statistics for serial correlation in the levels and squares, respectively, with 36 degrees of freedom. The value of the χ^2 distribution with 36 degrees of freedom and at 5% level of significance is 51.0.

Table 2. Maximum Likelihood Estimates of Full and Restricted GARCH Models for Corn under Full Vech Specification

Parameter	---- Full Model ----		-- Restricted Model --	
	Estimate	t-value	Estimate	t-value
δ_0	0.00571	4.020	0.00638	4.291
δ_1	-0.02319	-4.048	-0.02446	-5.307
ω_1	0.00033	3.845	0.00058	3.611
α_{11}	0.07473	7.154	0.13403	5.076
α_{12}	0.24601	5.965	0.19500	4.928
α_{13}	0.00010	0.086	0.00000	-0.001
β_{11}	0.68550	23.207	0.63503	11.466
β_{12}	0.01080	1.288	0.00007	0.027
β_{13}	0.00808	0.577	0.00020	0.051
ω_2	0.00056	5.273	0.00047	3.316
α_{21}	0.00941	1.347	0.00000	0.008
α_{22}	0.14858	4.405	0.15502	-3.625
α_{23}	0.12960	7.821	0.19536	4.205
β_{21}	0.00003	0.043	0.00000	0.007
β_{22}	0.43334	6.344	0.60973	7.883
β_{23}	0.23640	4.210	0.02664	0.410
ω_3	0.00025	2.170		
α_{31}	0.10230	8.577		
α_{32}	0.08588	2.209		
α_{33}	0.28604	7.488		
β_{31}	0.27077	3.877		
β_{32}	1.24999	6.839		
β_{33}	0.18512	1.257		
γ			0.86903	38.541
Q(36) Δp_t		45.64		46.50
Q(36) Δf_t		46.47		47.39
Q ² (36) Δp_t		24.51		23.96
Q ² (36) Δf_t		24.13		24.84
Log-likelihood		2490.17		2453.76

Notes: Q(36) and Q²(36) denote the Ljung-Box Portmanteau test statistics for serial correlation in the levels and squares, respectively, at 36 degrees of freedom.
The value of the χ^2 distribution with 36 (6) degrees of freedom and at 5% level of significance is 51.0 (12.6).

Table 3. Maximum Likelihood Estimates of Full and Restricted GARCH Models for Corn under Augmented Diagonal Specification

Parameter	----- Full Model -----		-- Restricted Model --	
	Estimate	t-value	Estimate	t-value
δ_0	0.00621	4.546	0.00591	3.512
δ_1	-0.01875	-3.693	-0.03152	-13.592
ω_1	0.00158	5.444	0.00123	16.498
α_{11}	0.36912	7.775	0.11747	7.402
β_{11}	0.37972	5.115	0.61719	163.782
ω_2	0.00084	5.717	0.00194	8.181
α_{22}	0.22674	9.731	0.03350	12.089
β_{33}	0.58691	13.991	0.48077	14.589
ω_3	0.00106	5.606		
α_{33}	0.26796	8.340		
β_{33}	0.51342	12.110		
γ	-0.04581	-1.365	0.76359	38.394
Q(36) Δp_t	51.62		49.31	
Q(36) Δf_t	43.14		49.80	
Q ² (36) Δp_t	29.29		50.05	
Q ² (36) Δf_t	28.31		220.57	
Log-likelihood	2465.08		2354.94	

Notes: Q(36), and Q²(36) denote the Ljung-Box Portmanteau test statistics for serial correlation in the levels and squares, respectively, at 36 degrees of freedom.
The value of the χ^2 distribution for 36 (3) degrees of freedom at 5% level of significance is 51.0 (7.8).

Figure1. Corn: Optimal Hedge Ratios

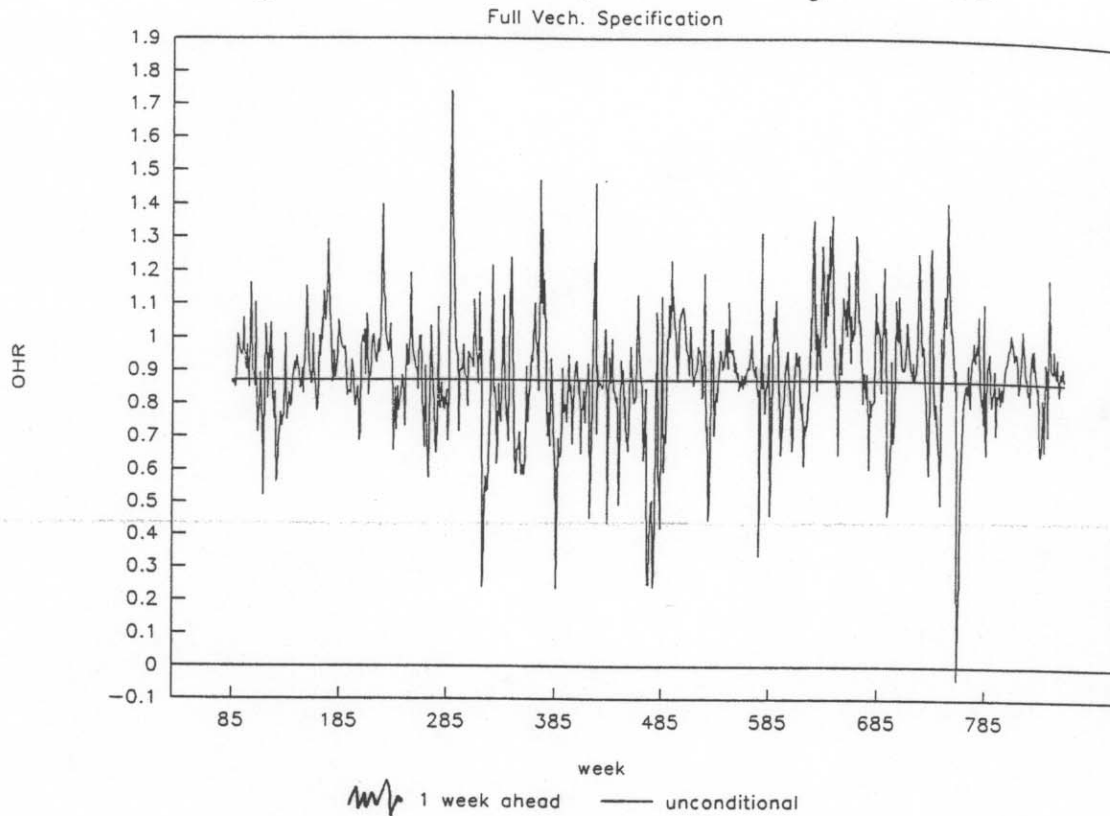


Figure2. Corn: Optimal Hedge Ratios

