Options Portfolios in the Presence of Non-Linear Risk

by

Kevin McNew

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Options on futures give hedgers a way to construct a risk management portfolio which has similar properties to the risk they face in the cash market. Of particular importance is the benefit that options provide when the cash position value is non-linearly related to the futures price. Such a situation is particularly prevalent for grain producers who face random cash prices and output. This study presents two methods for constructing an options portfolio composed of different strike prices. An empirical application for regional corn production in the U.S. demonstrates that both methods are similar in terms of risk reduction and, in some instances, provided significant improvements from using futures.

Introduction

It is well documented that when a firm's cash position value is linearly related to the futures price, a futures position is useful for reducing risk. The hedge ratio, which is one of the more popular statistics in academic futures market research, provides an estimate of the slope between the physical (cash) position value and the futures price. Thus, a firm which hedges in the proportion dictated by their hedge ratio will have changes in the physical position value match changes in the futures position value in an offsetting fashion. The result is the removal of all systematic risk, leaving only the undiversifiable basis risk (Peck).

When the physical position is not linear in the futures price, then options become a useful mechanism for hedging (Lapan, Moschini and Hanson; Sakong, Hayes and Hallam). Unlike the futures position value which is linear in the futures price, the non-linear payout structure of options allows a firm to construct a portfolio which more closely matches the true form of their risk in the physical market.

Under what conditions is the value of the physical position non-linear in the futures price? The physical position, which is the product of the spot price and output, can be non-linearly related to the futures price if either the spot price is non-linear in the futures price or if output is random. Both are prevalent for agricultural crop producers. In markets where spatial trade is important, interregional price arbitrage bounds can lead to disproportionate changes in prices. Thus, the relation between any two spatial prices is likely to be non-linear (McNew) indicating that firms spatially dispersed from the futures delivery site (or price) will likely face a non-linear relation between the spot price received and the futures price. When output is random, as in the case of crop production, the interaction of the random spot price and random output likely result in a non-linear relation between the physical position value and the futures price as was demonstrated by Sakong, Hayes and Hallam.

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The presence of non-linear risk implies that options are a useful mechanism for meeting the non-linear shape of the physical position. However, options present producers with a number of different strike prices from which to select an options portfolio. Until now, no guidance has been offered as to how to select a portfolio of option strike prices. In this study, a simple approach is presented which allows producers to determine the weight of various strike price options in their portfolio. Just as the hedge ratio gives an indication of how much to hedge using futures when risk is linear, the slope of the function between the physical position and futures price also guides the producer in selecting the portfolio composition.

Model

Consider a risk-averse agent which faces an uncertain value of a long cash position. To reduce the risk associated with this cash position, the agent can use options on futures. Denote the futures price on the option expiration date as F and the available strike prices as S_1, $S_2,\ldots,S_m$. A put option is a contingent claim which has positive value if the futures price at expiration is less than the strike price. Therefore, the value of the k^{th} put option is:

(1) $$v_k = max(0, S_k - F).$$

The premium for each of the m put options is denoted $\pi_1, \pi_2,\ldots,\pi_m$. Let $\alpha_1, \alpha_2,\ldots, \alpha_m$ denote the amount of each put option purchased by the agent. Denote as $\mathbf{v}$, π, and α as the mx1 vectors of the put option values, premium prices and put option positions, respectively. Using this notation, the options portfolio value can be expressed as:

(2) $$\bar{V} = \alpha'(\mathbf{v} - \pi)$$

Assume that the firm's risk in the cash or physical market can be represented by the relationship:

(3) $$\bar{C} = g(F) + u$$

where u is a serially independent error process which is independent of F and represents changes in supply and demand shocks to the agent's local market. Because this is a long cash position value then $g'(F)\geq 0$, i.e., the value of the cash position increases with higher prices. In full, the combination of the physical position and the options portfolio implies that the firm has the resulting risk still present denoted as:

(4) $$e(F) = g(F) + u + \bar{V}(F).$$

Because $g'(F)\geq 0$ and $\bar{V}'(F)\leq 0$ then the cash position and options portfolio are likely to have offsetting effects leading to more stability in the net position, $e(F)$.

The problem of the firm is to select the portfolio of options α which reduces the risk faced from fluctuations in the cash market value. To do so, a mean-variance framework is assumed which can be justified as a Taylor-series approximation to the popular expected utility methodology. Under the additional assumption that options prices are unbiased (i.e., $E(v_k)=\pi_k$ for all k) the mean-variance framework of this problem reduces to a variance minimization problem and is independent of the agent's degree of risk aversion.

The problem of choosing the portfolio composition under variance minimization can be expressed as:

$$(5) \quad \min_{\alpha} \quad Var[e(F)]/2 \quad = Var[g(F)]/2 + \alpha' Var(\mathbf{v})\alpha/2 + \alpha Cov[g(F),\mathbf{v}]$$

where Var() and Cov() are the variance and covariance operators. The optimal solution to this problem is:

$$(6) \quad \alpha^{*} = - [Var(\mathbf{v})]^{-1} Cov[g(F),\mathbf{v}].$$

When g() is linear and of the form $g(F)=\beta F$ then the solution reduces to:

$$(7) \quad \begin{array}{l} \alpha_k=0 \text{ for all } 1\leq k<m \\ \alpha_m=\beta. \end{array}$$

Substituting this solution into the net-profit function in (4) gives:

$$(8) \quad e(F)= \beta F+u+\beta[\max(S_m-F)-\pi].$$

For S_m sufficiently large, the only remaining risk in e(F) is the undiversifiable risk, u.[1] Thus, under linear risk only the largest strike price option is needed and the hedge ratio is equal to the slope of the relation between the cash position value and the futures price.

When g() is not linear, however, the optimal options portfolio is more complicated. Now, one must know the covariance matrix of the put option values as well as the covariance between the risky physical value and the put option values. Thus, not only is the stochastic structure of the futures price important but so is the shape of the physical risk function, g().

Although, the options portfolio in equation (6) is optimal in the sense of minimizing net-profit variability, it is also somewhat taxing to estimate and therefore implement. As an alternative, we consider a somewhat different problem. In this problem, the agent constructs a piecewise linear function (the options portfolio) which closely matches the non-linear risk function, g().

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[1] Alternatively, one could consider a short futures position in the portfolio which is a put option with infinite strike price. As long as the largest strike price is sufficiently large enough, there is no loss in generality by not considering a short futures position in this problem.

Formally, the problem can be specified as:

$$(9) \qquad \min_{\alpha} \int [g(F) - \bar{V}(F)] dF$$

$$\text{s.t., } g(F) \geq -\bar{V}(F).$$

This is a common methodology in linear seperable programming modeling (see e.g., Hadley).

The solution to this problem leads to an application of the mean-value theorem between any two strike prices. Denote the slope of g(F) between any two strike prices, s_k and s_{k+1}, as β_k where

$$(10) \qquad \beta_k = \frac{g(S_{k+1}) - g(S_k)}{S_{k+1} - S_k}$$

The options portfolio, which is a piecewise linear function, has a slope of $\alpha_k + \alpha_{k+1} + \ldots + \alpha_m$ when the futures prices is between the strike prices S_k and S_{k+1}. Thus, setting the slope of g() equal to the slope of the options portfolio value gives m-equations each satisfying:

$$(11) \qquad \beta_k = \sum_{i=k}^{m} \alpha_k \ .$$

Solving for the options portfolio yields:

$$(12) \qquad \alpha_k = \beta_k - \beta_{k+1} \text{ for all } k < m$$
$$\alpha_m = \beta_m.$$

Thus, the largest strike price option weight is equal to the slope of the risk function at the highest strike price level. The remaining portfolio weights are based on the difference in consecutive slope values and are related to the second derivative of g().

For crop producers, they often face negative correlation between their local cash price and crop yield. Also, local cash prices tend to be positively related to the futures price. In this type of situation, the risk function g() is likely to be concave in the futures price. This implies that the option portfolio weights increase as the strike price decreases. This principle and the relationship between the slope of the risk function and portfolio weights is illustrated in figure 1.

Because only the slope of the risk function is required, this procedure is considerably simpler to implement then the variance minimization rule presented above. However, whether there is a significant loss in risk protection from using this strategy is unclear. The remainder of this study is devoted to answering this question empirically for hypothetical corn producers from various regions of the United States.

Data and Empirical Methodology

To determine whether options provide would be beneficial to individual corn producers, one would prefer to have a long history of producer prices and yields. However, such disaggregated data are difficult if not impossible to procure. Thus, this study employs corn yield data for the largest corn producing county in each of the top ten corn producing states in the United States based on total 1994 corn production. Twenty-two years of crop yields, from 1973-1994, for these ten counties were collected and matched with state-average cash prices in November which coincides roughly with the harvest period. The ten states are reported in table 1 along with each county used for a state.

It is assumed that the options are purchased in May coinciding with the spring planting time. Thus, the average of the December Corn futures price in May is computed, which is denoted as f. Also, the average December Corn futures price in November (F) is used to construct the intrinsic value of the price options, **v**.

Because only 22 years of data are available, a joint distribution of the random variables is assumed and its parameters estimated for simulation purposes. From this distribution, simulation analysis is used to generate a large number of observations and perform the analysis. Based on the individual county yields (y), the state-average cash prices (p) and the futures prices in May (f) and November (F), the joint distribution of these random variables was estimated for 1994 assuming joint log-normality. Log-normality is used because it allows for skewness and non-negative random variables. To construct the mean of the price random variables, it is assumed that futures prices are unbiased so that the expected value of the logged futures price in November is the log of the futures price in May. The expected log cash price is equal to the log of the futures price in May plus the average basis over the sample. For yields, each yield series was regressed on a time trend and used to estimate the expected value of log-yields for 1994. In full, the distribution assumptions imply:

$$\begin{pmatrix} \ln(p) \\ \ln(F) \\ \ln(y) \end{pmatrix} \sim N\left[\begin{pmatrix} \ln(f+b) \\ \ln(f) \\ \ln(\delta_0+\delta_1 t) \end{pmatrix}, \Sigma\right] \qquad (13)$$

where b is the average basis for each state, t is a time trend and Σ is the covariance matrix of the three random variables.

Three different strike levels are considered for the price options. These three values coincide with the current mean of the futures price, one standard deviation below the mean, and one standard deviation above the mean. The current price for the December 1996 futures contract (in April 1996) is around 318 so, for concreteness, the mean futures price is assumed to be 300. Thus, the three strike prices are $S_1=260$, $S_2=300$ and $S_3=340$. Although more strike levels could be used, a preliminary analysis indicated that when the number of strike levels is large, the optimal option positions are so small that it would be impractical because of the fixed contract size (e.g., 5,000 bushels for price options).

Based on the joint distribution of the random variables for 1994, 1,000 samples of 1,000 observations were generated for ln(p), ln(F), and ln(y). After making the transformation to non-logged values, they were used to construct the cash value (C), the option values (v). The optimal value of α is estimated via a linear regression of the form:

(14) $$C=\mu + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + e$$

where v_1, v_2 and v_3 are the intrinsic value for options with strike prices 260, 300, and 340, respectively. The 1,000 values of α, when averaged, give sample estimates of the variance minimization portfolio described by (6). Hereafter, this solution will be denoted α^v to denote the options portfolio which minimizes the variance of net-profits.

The second procedure makes use of the shape the risk function g(F) which describes the relationship between the futures price and producer revenue. Although the shape of g() is unknown, it is likely to be non-linear and concave in the futures price for reasons discussed in the previous section. Thus, a model of the form

(15) $$\ln(C) = \beta_0 + \beta_1 \ln(F) + \beta_2\{\ln(F)\}^2 + e,$$

is estimated. This model, which is quadratic in the log of futures price, appears appropriate for all states based on some preliminary analysis for other specifications. The exception was Kansas where it was determined that the hypothesis $\beta_2=0$ could not be rejected. Thus, for Kansas, the g() function was assumed to be log-linear.

Using the estimated model for each state, the option portfolio weights were determined by the slope of g() in between various strike prices. The option portfolio weights based on the slope of g() are denoted as α^g.

Results

Option portfolio weights are given in table 2. Both the portfolio for the variance minimization problem (α^v) and the portfolio based on the slope of the risk function (α^g) are given in normalized form for each state. These values represent the amount of each option to use for each one bushel in expected yield and are comparable to hedge ratios based on the use of short futures positions. As a means of comparison, the hedge ratio based on a linear regression of C on F is also given in table 2.

In general, the fraction of output hedged either with a futures position or a set of put options is relatively small. For example, when using futures, the proportion of output hedged ranges from 0.18 to 0.43. These low values are the result of significant yield variation. When using options, the sum of the option positions ranges from 0.47 to 0.67 for the variance minimization case. Although still far away from fully hedged, options do allow producers to protect more of their expected output by being able to account for the non-linear relationship between revenue and the futures price.

Option weights certainly vary across states. For some states—like Illinois and Minnesota—the portfolio is composed mostly of the low strike price option with higher strike prices having lower weights. Such a result is indicative of a great degree of concavity in the risk function. For Indiana, Iowa and Missouri, the portfolio weights are largest in the highest and lowest strike price with little use of the strike price at the mean. Although these states do gain some benefit from being able to account for the concave relationship, it is certainly less distinct than for Illinois and Minnesota. The remaining states sell the middle strike price option (i.e., negative hedge ratio). This portfolio is similar to the classic 'bear spread' (i.e., buy a high strike price put option, sell a low strike price put option) option strategy although the relative weights dictate a somewhat different payout. Because the weight on the high strike price options dominates that of the low strike price option, the result is a profit that is decreasing and less convex in the futures price. As such, these states have nearly linear risk in the futures price.

Comparing the option portfolios from the two optimization criteria show that both lead to very similar option portfolios—particularly for Illinois, Indiana, Iowa, Minnesota, and Missouri. However, to adequately determine how comparable to each other the two strategies are, it is important to identify the risk reduction potential of each. To do so, the standard deviation of net-profit for both option strategies and the futures hedge strategy were computed. These measures of risk were compared to the unhedged revenue risk to determine how much risk is removed from the base of no hedging for each strategy. These results are given in table 3.

Overall, the risk reduction benefits of using options or futures are relatively small. This is because of the uncontrollable yield variability. Even though some states do have some negative correlation between yield and price, it is less than perfect so that futures or options on price alone remove only a small fraction of total revenue risk. However, in comparing across strategies, it is obvious that most states do get higher risk reduction potential from using options over futures. Indeed, even the simple rule for option weights based on the risk function gives nearly similar risk reduction benefits to the variance minimization rule. For the states of Kansas, Nebraska, South Dakota, and Wisconsin, the option portfolio from the risk function is not as successful at reducing risk as a short futures position.

Final Remarks

Options have many features that producers prefer for hedging. The ability to limit your loss and set a price floor for the product are commonly listed reasons why options are preferred to futures for risk management purposes. Additionally, crop farmers, who can face significant production risk, can use options to help reduce some of the non-linear risk they face between yield and price.

Using a variance minimization rule and a simple rule based on the shape of the risk function between revenue and the futures price, it was shown that both can lead to similar risk reduction potential. In many regions of the country, it was shown that either strategy reduces more risk than using futures.

Although options did remove some of the revenue risk, the strategies presented here fell well short of removing a large portion of it. This is largely due to the risk not being perfectly correlated with price. With the innovation of corn yield futures and options contracts at the Chicago Board of Trade, significant improvements may exist in achieving lower risk levels for corn farmer revenue by incorporating them in the hedging portfolio.

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Figure 1. Relationship Between the Risk Function Slope (β) and the Option Portfolio Weights (α).

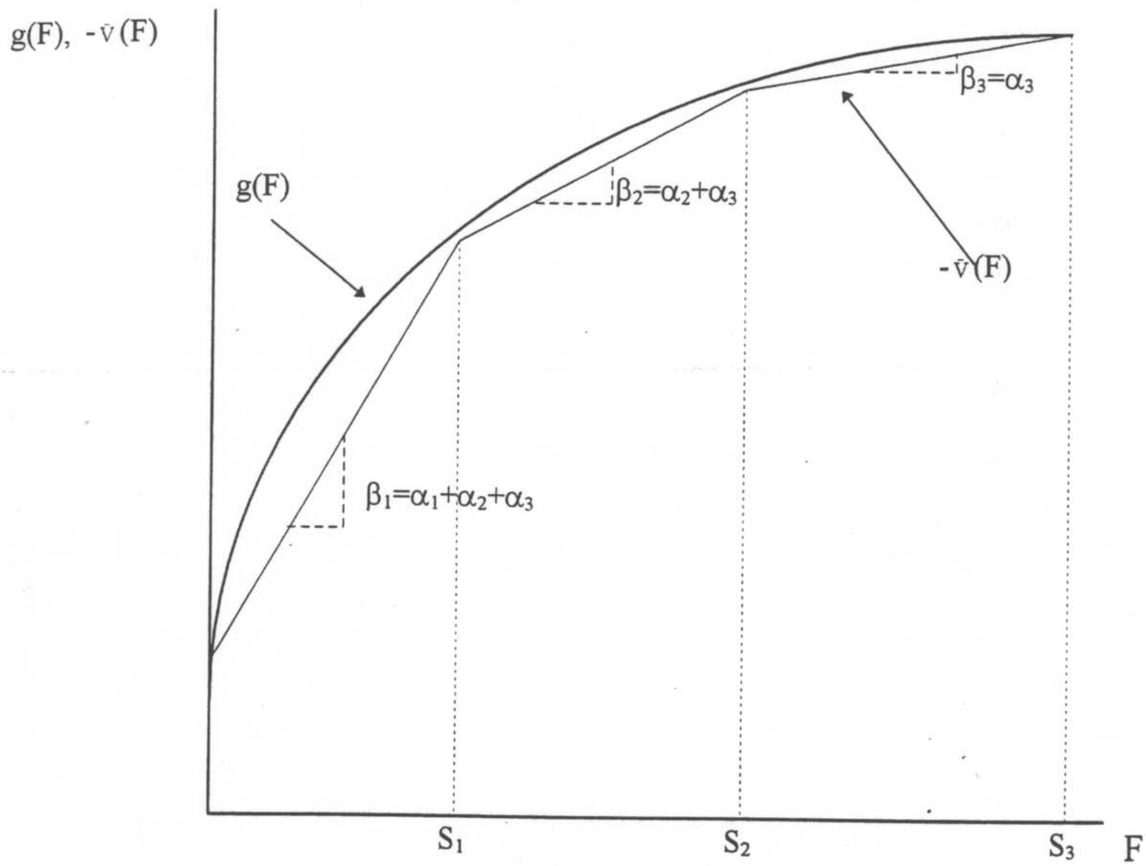

Table 1. Top Ten Corn Producing States and Top Corn Producing County in Each State.

State	County
Iowa	Kossuth
Illinois	McLean
Nebraska	Phelps
Minnesota	Renville
Indiana	Jasper
Ohio	Putnam
Wisconsin	Dane
South Dakota	Minnehaha
Kansas	Sherman
Missouri	Atchison

Table 2. Option Portfolio Weights Based on Variance Minimization, the Slope of the Risk Function and Hedge Ratios Using a Short Futures Position.

State	α^v_1	α^v_2	α^v_3	α^g_1	α^g_2	α^g_3	Hedge Ratio
Illinois	0.25[1]	0.18	0.05	0.20	0.11	0.05	0.18
Indiana	0.26	0.05	0.28	0.18	0.11	0.18	0.30
Iowa	0.22	0.01	0.29	0.15	0.09	0.18	0.28
Kansas	0.18	-0.22	0.58	0.05	0.03	0.35	0.40
Minnesota	0.24	0.17	0.06	0.20	0.11	0.05	0.18
Missouri	0.29	0.10	0.24	0.22	0.13	0.16	0.30
Nebraska	0.21	-0.15	0.53	0.12	0.07	0.30	0.40
Ohio	0.23	-0.05	0.40	0.15	0.09	0.24	0.34
South Dakota	0.25	-0.14	0.56	0.14	0.09	0.33	0.43
Wisconsin	0.22	-0.10	0.46	0.13	0.08	0.27	0.37

1. All values represent the proportion of expected yield to hold in each strike price. α^v denotes the variance minimization rule and α^g represent the option weights based on the risk function. Strike prices are 260, 300 and 340 for subscripts 1, 2 and 3.

Table 3. Percent Reduction in the Standard Deviation of Revenue For Each Strategy from Unhedged Position.

State	Options Based on Variance Minimization	Options Based on Risk Function	Using Futures
Illinois	2.5	2.4	1.6
Indiana	7.2	6.8	6.1
Iowa	4.8	4.5	4.2
Kansas	7.0	6.6	7.0
Minnesota	2.0	1.9	1.3
Missouri	4.4	4.2	3.5
Nebraska	15.1	14.3	14.7
Ohio	7.7	7.3	7.1
South Dakota	17.8	16.9	17.1
Wisconsin	8.9	8.4	8.4