Non-Expected Utility Theories:

Weighted Expected, Rank Dependent,

and, Cumulative Prospect Theory Utility

by

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This paper discusses some of the failings of expected utility including the Allais paradox and expected utility's inadequate one dimensional characterization of risk. Three alternatives to expected utility are discussed at length; weighted expected utility, rank dependent utility, and cumulative prospect theory. Each alternative is capable of explaining Allais paradox type problems and permits more sophisticated multi dimensional risk preferences.

Keywords: Expected utility, weighted expected utility, rank dependent utility, cumulative prospect theory, Allais paradox

Introduction

Expected utility is the standard model for analyzing decision making under uncertainty. Expected utility is well known to economists and is relatively easy to work with, most in the field are not even aware of the alternatives. Despite it's popularity there are problems with expected utility that suggest it is not a suitable model of human behavior. Expected utility suffers from Allais paradox type problems that suggest, experimentally, that agents weight payoffs AND probabilities in decision making. Also, the manner in which risk is characterized in expected utility leads to absurd conclusions, this was recently pointed out by Rabin and Thaler (2000). In light of these shortcomings with expected utility we present three alternatives that mitigate the problems mentioned above. The alternatives are weighted expected utility, rank dependent utility, and cumulative prospect theory.

Section 1: Expected Utility
1.1 Expected Utility

Expected utility is characterized by

(1) $u(L) = p_1 u(x_1) + (1-p_1)u(x_2)$.

Where L is a lottery that pays prize x_1 with probability p_1 and x_2 with probability p_2. The utility of the lottery is the expected utility of the prizes. von Neumann and Morgenstern (1944) show that expected utility can be derived from a series of seemingly innocuous axioms.

Consider a lottery that pays L_1 with probability p and pays L_2 with probability 1-p, where L_1 and L_2 are themselves lotteries. The axioms are as follows:

A.1:Completeness: For any two sets of lotteries L_1 and L_2, $L_1 \succeq L_2$ or $L_2 \succeq L_1$ or $L_1 \sim L_2$.

A.2: Transitivity: For any three sets of lotteries if $L_1 \succeq L_2$ and $L_2 \succeq L_3$ then $L_1 \succeq L_3$. The first two axioms permit the existence of lotteries and preferences. That is, they ensure that an agent can specify which states of nature he prefers and that these preferences are logically consistent. Together A.1 and A.2 imply that preferences can be ordered.

A.3: Continuity: For three sets of lotteries such that $L_1 \succeq L_2 \succeq L_3$ there exists some p such that $(L_1, p; L_3, 1\text{-}p){\sim}L_2$, the right hand side of this relation is a compound lottery (or mixture) that results in L_1 with probability p and L_2 with probability 1-p.

The continuity axiom is necessary to ensure that agents' preferences can be modeled with a continuous function.

A.4: Independence: If $L_1 \succeq L_2$ then $(L_1, p; L_3, 1\text{-}p) \succeq (L_2, p; L_3, 1\text{-}p)$.

The final axiom, A.4, is the most important and controversial of the axioms. Without the independence axiom it would not be possible to invoke the most appealing characteristic of expected utility, the utility of a lottery is the expected utility of the lottery's payoffs.

One characteristic of (1) is that it is linear in probabilities. It is this characteristic that is questioned by skeptics of expected utility. Another implication of (1) is that the expected utility function is monotonic, which in turn implies stochastic dominance preference. Agents are better off by increasing the probability of high payoff events, this is the probabilistic counterpart to the "more is better" implication of traditional utility functions (Machina, 1987). Expected utility also *seems* to permit a wide range of attitudes towards risk. If u(.) is a concave function then the agent is risk averse; agents prefer a sure payoff to a lottery with equal expected value. Expected utility also permits risk neutral and risk seeking agents. Concave u(.), which are typically employed, exhibit diminishing marginal utility for money.
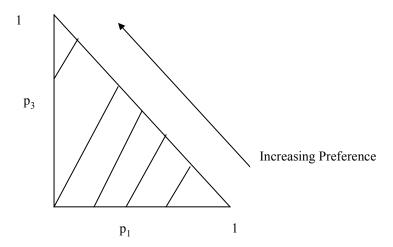
1.2 Triangle Diagrams

Before discussing the problems with expected utility and the prominent alternatives a graphical explication of expected utility will be presented using triangle diagrams. This method of analysis will be useful for comparing the non-expected utility theories to expected utility. Marschak (1950) pioneered the use of triangle diagrams although it was Machina (1982) who made widespread use of them.

Consider a lottery with three outcomes ordered lowest to highest x_1, x_2, and x_3 with associated probabilities p_1, p_2, and p_3. The probabilities sum to one. Although this is a three dimensional lottery it can be depicted in a two dimensional diagram (below). This can be accomplished because $p_2 = 1 - p_1 - p_3$. By solving,

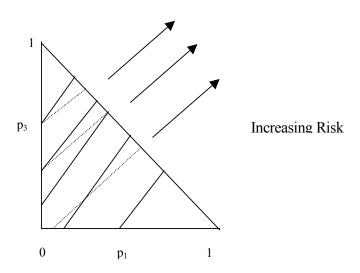$$(2) \qquad \bar{u} = \sum_{i=1}^{3} u(x_i)p_i = u(x_1)p_1 + u(x_2)(1 - p_1 - p_3) + u(x_3)p_3 ,$$

where $\bar{u}$ is a constant, for different $\bar{u}$ indifference curves can be generated. The solutions to (3) form parallel straight lines in p_1-p_3 space.

Figure 1: Triangle Diagram with Expected Utility Indifference Curves

Movements along the vertical axis increase the likelihood of x_3 at the expense of x_2. Movements along the horizontal axis increase the likelihood x_1 at the expense of x_2. At the origin the probability of observing x_1 and x_3 is zero so the probability of observing the middle outcome, x_2, is one. Expected utility indifference curves are straight, parallel lines in the triangle diagram with movement to the North-West yielding greater utility, i.e. the probability of the large payoff is higher at the expense of the low payoff (Machina, 1987).

Figure 2: Triangle Diagram with Expected Utility Indifference Curves and iso-Expected Value lines.

In Figure 2 the triangle diagram is augmented with iso-expected value lines (dotted). Along a given iso-expected value line the expected payoff of a prospect does not change. However moving along the line in a North-East direction increases the probability of x_3 at

the expense of x_2, so we increase the risk of the prospect. The points on a given iso-expected value line represent a mean preserving spread or a pure increase in risk.

In figure 2 notice that the expected utility indifference curves are steeper than the iso-expected value lines. If we move along a iso-expected value line in the North-East direction we cross indifference curves that are progressively lower as we move. As we increase the risk of a gamble, we lower the utility of the gambler. Therefore the indifference curves shown here are for a risk averse agent. A risk seeking agent would have indifference curves that are flatter then the iso-expected value lines.

Section 2: Problems With Expected Utility
2.1 Allais Paradox

Violations of expected utility were first widely recognized as a result of work by Allais (1953). The Allais Paradox is based on empirical observations that imply agents weigh both expected outcomes and the probabilities associated with expected outcomes. Weighting probabilities is a clear violation of expected utility theory, which requires that expected utility functions be linear in probabilities.

Consider the following situation. You, as the agent, must choose between hypothetical lotteries. There are two different choice sets; for each choice set there are two lotteries from which you can choose. In each lottery there are three payoffs and their corresponding probabilities. (Note, m denotes million.)

Choice 1
lottery1: A = ($5 m, 0; $1 m, 1; $0, 0) **OR** B = ($5 m, .10; $1 m, .89; $0, .01)

Choice 2
lottery 2: A= ($5 m, 0; $1 m, .11; $0, .89) **OR** B = ($5 m, .10; $1 m, 0; $0, .90)
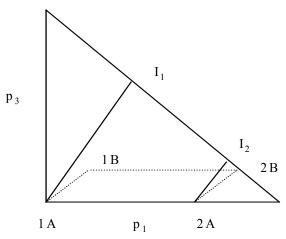
For example, in lottery 1A you are guaranteed a payoff of $1 million and there is zero probability of winning $5 million or $0. In lottery 1B there is a .10 probability of winning $5 million, a .89 probability of winning $1 million, and a .01 probability of winning $0. You must choose between 1A and 1B, and between 2A and 2B. It may be helpful to note your decision.

Many, if not most, agents prefer lottery 1A to lottery 1B and prefer lottery 2B to lottery 2A. This empirical tendency directly contradicts expected utility theory. Specifically, the independence axiom is violated. If you are an expected utility maximizer then you must either prefer 1A to 1B and 2A to 2B, or 1B to 1A and 2B to 2A. Agents may prefer 1A to 1B because they like the idea of becoming a millionaire with certainty, implying risk aversion. But in choice set 2 the gambles are quite different with a high probability in each lottery of not winning any money. So, the agent may simply say, "what the heck, I'll just go for 2B because my chance of winning $5 million is very similar to my chance of winning $1 million and $5 million is much more." Unfortunately for expected utility, this plausible line of reasoning violates the independence axiom. Basically the typical

agent responds in a more risk averse manner in choice set 1 and more risk neutral in choice set 2.

Below is a triangle diagram depicting the Allais Paradox.

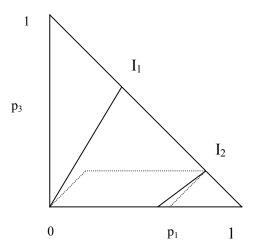Figure 3: Allais Paradox Triangle Diagram

The four gambles are shown. If an agent prefers gamble 1A to 1B then that suggests that his indifference curves are relatively steep. Note the steep indifference curve through 1A. If you prefer 1A to 1B then your expected utility indifference curve is similar to I_1. However, I_2 is parallel to I_1, as required by expected utility theory, but I_2 forces the agent to prefer 2A to 2B. Preferring 2B to 2A violates expected utility. Simply put, it is not possible to generate expected utility indifference curves that support most agents' preferences over the gambles. For the indifference curve to accurately represent most agents' preferences, line I_2 must be flatter than I_1. This is called "fanning out."

The Allais paradox exists, in all likelihood, because agents place weights on the probabilities of expected outcomes. The weighting of probabilities is intuitively linked to individual's attitudes towards risk. It seems plausible that people weight *catastrophic* (low probability) losses or gains (as in the Allais paradox) differently than they weight *normal* losses or gains. What constitutes *catastrophic* and *normal* is difficult to characterize mathematically. Because this type of behavior is so prevalent researchers have attempted to develop new utility theories that account for the Allais paradox.

Machina (1987) argues that many violations of expected utility, including the Allais Paradox, exist because indifference curves fan out for most agents. Basically fanning out requires that indifference curves become steeper as they move from the lower right to the upper left of the triangle diagram (Figure 4).

Figure 4: Fanning Out

The Allais paradox is not a paradox at all for agents with indifference curves I_1 and I_2, although it is not possible to generate these indifference curves with expected utility.

There are numerous examples of Allais paradox type violations of expected utility. The empirical evidence is typically gathered through experiments in which participants choose between choice sets similar to the examples presented here. Some of the notable evidence can be found in Allais (1953,1979), Camerer (1989), Ellsberg (1961), and Kahneman and Tversky (1979). Interestingly it has been shown that animals also violate the axioms of expected utility (Battalio, Kagel, and Mac Donald, 1985).

The most obvious way to reconcile the Allais paradox and many other problems with expected utility is to allow the agent to weight the probabilities associated with the relevant outcomes. Violations of expected utility are the result of requiring indifference curves to be linear and parallel in probability space. In Section 3 we discuss some of the various ways researchers have attempted to loosen this rigid requirement.

2.2 Risk Aversion

In expected utility the attitude of the agent towards risk depends entirely on the curvature of the valuation function. Preference for risk is quantified by the coefficient of absolute risk aversion, often referred to as the Arrow-Pratt measure which was developed independently by Pratt (1964) and Arrow (1965). The measure is based on the first and second derivatives of the utility function with respect to the choice variable. The measure is,

$$(3) \qquad r(x) = \frac{-u''(x)}{u'(x)},$$

the negative of the second derivative divided by the first derivative. The second derivative alone determines if the function is concave and hence the agent risk averse. But by normalizing the measure by dividing by the first derivative the measure is invariant to changes in scale. A positive r(x) corresponds to a risk averse agent.

Rabin and Thaler (2001) make a compelling argument that expected utility takes too simplistic a view towards risk. Agents' risk attitudes are determined entirely by the shape, i.e. concavity, of the utility function. Suppose there is a risk averse agent that rejects a 50-50 gamble with payoffs of $11 and -$10. Now the agent is offered a 50-50 bet with payoffs of $x and -$100, how large does x have to be for the agent to accept the bet? Using a theory developed by Rabin (2000) they show that this agent will not accept the second bet no matter how large x is for *any initial level of wealth*. This is obviously ludicrous although plausible under expected utility and its definition of risk aversion. Basically, the agent's aversion to the small bet suggests that his marginal utility for wealth must decline very rapidly; hence, he will not participate in the second bet because he gains so little utility for additions to wealth. The nonsensical results are caused by expected utility's dependence on a single measure of risk aversion that fails to allow certain plausible behavior. This deficiency of expected utility causes Rabin and Thaler to question why economists are so attached to the paradigm.

Rabin (2000) shows that by only assuming increasing and concave utility functions that agents will behave absurdly to large stakes bets if they act risk averse to more moderate bets. So if we use expected utility to model agents in situations where the payoffs are non negligible but small we are necessarily implying that these agents act in absurd ways if the stakes are large.

In the following section we will show that the alternatives to expected utility do not rely on a single risk aversion measure. The alternatives to expected utility we will discuss have two or more ways of characterizing agent's risk preferences.

Section 3: Alternatives to Expected Utility
3.1 Introduction

The most obvious way to improve expected utility is to relax the independence axiom and allow the probabilities to enter the function in a nonlinear manner. If probabilities are allowed to be nonlinear then the agents' indifference curves need not be linear nor parallel. In the remainder of this section five non-expected utility theories will be discussed, of which three appear to be promising alternatives to expected utility: weighted expected utility, rank dependent utility, and cumulative prospect theory. Two other non-expected utility theories, decision weighted utility and prospect theory, will be discussed because they represent important intermediate theories that led to rank dependent utility and cumulative prospect theory respectively.

3.2 Weighted Expected Utility
Weighted expected utility, sometimes referred to as weighted linear utility, is a tractable generalization of expected utility that permits Allais paradox type behavior. A weighted

expected (WEU) utility function was developed by Chew and MacCrimmon (1979a, 1979b) and Fishburn (1983). The functional form of WEU is as follows,

$$(4) \qquad V(L) = \frac{\sum_i u(x_i)p_i}{\sum_i w(x_i)p_i},$$

where u(.) is the valuation function, p_i is the probability of observing x_i, and w(.) is a weighting function. Weighted expected utility reduces to expected utility if w(.) is a constant.

The primary axiomatic difference between expected utility and weighted expected utility is that the latter has a weakened independence axiom. This axiom was developed by Chew and Waller (1986) the interpretation below is by Camerer (1989):

A.4.2 Weak Independence: For each probability p' there is another probability p'' for which $L_1 \sim L_2$ implies $p' L_1 + (1 - p')L_3 \sim p'' L_2 + (1 - p'')L_3$.

The strong independence axiom of expected utility forces $p' = p''$, but in weak independence they are not necessarily equal allowing the agent a certain amount of latitude in his preferences. The most important implication is that an agents' preferences are not necessarily "irrational" if the agent considers common consequences in stating preferences. The weighted expected utility function (4) satisfies the weakened independence axiom.
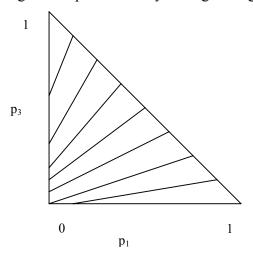
Recall equation (1), which defines the utility of a lottery as the expected utility of the payoffs. A similar but more complex statement can be made for weighted expected utility,

$$(5) \qquad V(L) = \frac{p_1 w(x_1)u(x_1) + (1 - p_1)w(x_2)u(x_2)}{p_1 w(x_1) + (1 - p_1)w(x_2)}$$

The numerator of (5) is similar to the expected utility function (1) except that the probabilities are weighted by the weighting function, w(.). The denominator of (5) normalizes V(.) by the expected weighting function. Collectively the numerator and denominator of (5) permit a non-linear weighting of probabilities.

Below (Figure 5) is a triangle diagram that depicts weighted expected utility.

Figure 5: Weighted Expected Utility Triangle Diagram

Note that the indifference curves are linear but fan out thus permitting Allais paradox type behavior. Because the indifference curves are not parallel they must intersect at some point. Weighted expected utility indifference curves all intersect at a single point called the hub, this point of intersection can lie inside or outside of (p_1,p_3) space.

By combining the Allais paradox triangle diagram with a weighted expected utility diagram it is possible to show how there is no paradox in weighted expected utility. This can be seen graphically in figure 4; the indifference curves I_1 and I_2 could have been generated by a weighted expected utility model.

In expected utility the agent's attitude toward risk is measured by (3) the Arrow-Pratt measure of absolute risk aversion. For weighted expected utility there are two measures used to characterize risk: sensitivity and eccentricity. These measures were developed by Hess and Holthausen (1990) and this discussion is based on their work. Suppose an agent has a wealth level of $\bar{x}$ and an additional uncertain source of wealth, e, with expected value μ and variance σ^2. Define the risk premium, Π, needed to make the agent indifferent between paying the premium and avoiding the risk implied by the uncertain source of wealth,

$$(6) \quad \frac{u(\bar{x}+\mu-\Pi)}{w(\bar{x}+\mu-\Pi)} = \frac{E[u(\bar{x}+e)]}{E[w(\bar{x}+e)]}.$$

Hess and Holthausen then take first order Taylor series expansions of u(.) and w(.) on the left side in μ-Π, and they take second order Taylor expansions of the left side in e. Solving for Π,

$$(7) \quad \Pi = \frac{S + \mu E}{(\dfrac{2}{\sigma^2 + \mu^2}) + E}$$

$$(8) \quad S = \frac{uw'' - u''w}{u'w - w'u}$$

$$(9) \quad E = \frac{u'w'' - u''w'}{u'w - w'u}$$

where the arguments in the functions u(.) and w(.) have been suppressed. Chew (1983) defines (8) as the measure of absolute risk sensitivity. Note that if w(.) is a constant then (8) reduces to the Arrow-Pratt measure. Unlike expected utility there is an additional measure of risk when w(.) is not constant, E. Hess and Holthausen (1990) define E as eccentricity, which measures the extent to which an agent deviates from expected utility. If an agent has zero eccentricity then he is an expected utility maximizer. Note that the more sensitive the agent the higher the premium must be to accept the risk. However, eccentricity is in the numerator and denominator so it is not as obvious how eccentricity affects the risk premium.

3.3 Decision Weighted Utility

There are other plausible means of constructing utility functions that are nonlinear in probabilities. Consider the following utility function,

$$(10) \quad V(L) = \sum_i \pi(p_i) u(x_i),$$

where π is a probability weighting function such that $\pi(1) = 1$ and $\pi(0) = 0$. The decision weighted utility approach is based on work by Edwards (1955, 1962), with subsequent refinement by Handa (1977). By modeling decision weights with $\pi(p_i)$ it is possible to reflect a wide range of preferences, based on the choice of $\pi(p_i)$, which accounts for some of the empirical contradictions found with expected utility.

However, this model has a major flaw initially pointed out by Fishburn (1978). The only choice of $\pi(.)$ that will maintain general monotonicity is to set $\pi(p_i) = p_i$, if this is done then (6) reduces to an expected utility function. Because of the flaw detected by Fishburn decision weighted utility is considered a research dead end, but its development and subsequent failure did lead to a more satisfying but different approach, rank dependent utility.

3.4 Rank Dependent Utility

Rank dependent utility functions, developed by Quiggin (1982), look similar to decision weighted utility functions at first glance. As with all of the non-expected utility theories, rank dependent utility permits nonlinear weighting of probabilities. A slight adjustment

to the definition of the relevant gambles is needed. The gamble has the form $(x_i; p_i)$ for i = 1 to n where the vector of payoffs and associated probabilities are sorted in order from lowest to highest.   In rank dependent utility the value of an outcome depends on the probability of realizing that outcome and the ranking of the outcome relative to the other possible outcomes. The rank dependent utility function is as follows,

$$(11) \qquad V(L) = \sum_i w_i u(x_i)$$

$$(12) \qquad w_i = \pi(p_i + ... + p_n) - \pi(p_{i+1} + ... + p_n)$$
$$w_n = \pi(p_n)$$

where $w_i(.)$ is a weighting function and $\pi(.)$ is a transformation of cumulative probabilities, mapping $[0,1]$ into $[0,1]$. The value $\pi(p_i + ... + p_n)$ is the subjective weight attached to the probability of getting a payoff at least as good as x_i, while the value $\pi(p_{i+1} + ... + p_n)$ is the subjective weight attached to the probability of receiving a payoff strictly better then x_i. The net result of these transformations is that the weighting function preserves monotonicity of $V(.)$. The valuation functions, $u(.)$, are typically the same as those used in expected utility, strictly increasing and concave.
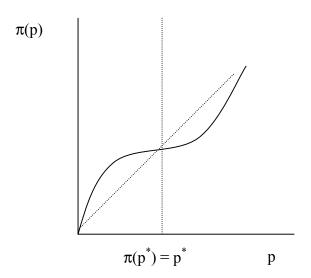
The use of cumulative probabilities in the manner specified by (12) has an appealing property relative to decision weighted utility models. Suppose there is an outcome with probability p_j; in decision weighted models the weight attached to this outcome will always be the same, but in rank dependent utility, by virtue of (12), the weight will reflect how "good" or "bad" a particular outcome is relative to the other outcomes (Starmer, 2000).   This is an intuitively pleasing property; agents weight the value of a payoff relative to the other payoffs available. Under some circumstances a payoff x_j with probability p_j may seem appealing compared to other payoffs, but under different circumstances the prospect (x_j, p_j) may be a unappealing payoff. How the agent weights the prospect depends on how it is ranked relative to the other prospects.

Obviously the choice of $\pi(.)$ is critical in specifying a rank dependent utility function. If $\pi(.)$ is concave then $\pi(p) \leq p$ for all p, this implies pessimism (Quiggin, 1993). A pessimistic agent over-weights lower ranked outcomes and under-weights higher ranked outcomes. So a pessimistic agent is more worried about bad states of nature, hence he places greater weight on relatively worse outcomes. An agent with a convex $\pi(.)$ is optimistic, he over-weights the higher ranked outcomes.

When discussing risk in rank dependent utility two factors must be kept in mind, the shape of the utility function (concavity implies risk aversion) and what type of transformation function is used. A pessimistic agent with a concave utility function will behave in a universally risk averse manner. A risk seeking agent, with convex $u(.)$, that is sufficiently pessimistic will also behave in a risk averse manner. In this way rank dependent utility is similar to weighted expected utility, concave utility functions imply risk aversion but the shape of the utility function alone does not dictate an agent's risk preferences.

Risk becomes slightly more complicated when one considers inverted S shaped (hereafter referred to as S shaped) transformation functions. An S shaped function necessarily crosses the 45 degree line, as in figure 7.
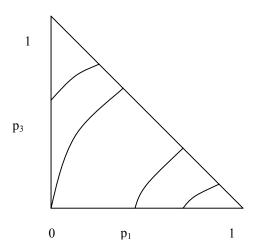
Figure 7: S Shaped Transformation Function

$$\pi(p^*) = p^*$$

At $\pi(p^*) = p^*$ the probability is not distorted; below p^* the agent is pessimistic and above p^* the agent is optimistic. The logic behind the S shaped transformation is that agents will over-weight low probability events and under-weight high probability events. The agents worry more about the more obscure, rarer, states of nature. This characteristic is intuitively pleasing; agents place greater weight on the less likely events for which they are less experienced and consequently more anxious. The S shaped transformation is favored by Quiggin and has found empirical support (Prelec, 1998). Quiggin favors an S shaped curve with $p^* = .50$. This property permits the observed violations of expected utility and it has the appealing property that 50-50 gambles are not affected by probability weighting (Stramer, 2000). Prelec (1998) finds that the p^* is closer to 1/3. In any event convex and S shaped cumulative probability transformations allow rank dependent utility to explain many violations of expected utility including the Allais paradox. If the transformation function is linear with slope one, the 45 degree line, then rank dependent utility reduces to expected utility.

In terms of triangle diagrams rank dependent utility functions appear as in Figure 8 (Camerer, 1989).

Figure 8:Rank Dependent Utility Triangle Diagram

The indifference curves are concave and parallel on the hypotenuse of the triangle and fan out West to East. The indifference curves become less steep and fan in from the South to North. Unlike weighted expected utility the curves are not parallel; however, they are capable of explaining Allais paradox behavior.

The axioms supporting rank dependent utility are essentially the same as the axioms for expected utility except for the independence axiom. The rank dependent utility analog to the independence axiom is called co-monotonic independence (Wakker, Erev, and Weber, 1994). Basically, co-monotonic independence asserts that preferences do not depend on common consequences unless the common consequence, substituting one common consequence for another, alters the rankings of the outcomes.

Expected utility is embodied by a relationship that is intuitively appealing - the utility of a gamble is the expected utility of the gamble's payoffs. Weighted expected utility has a similar but less aesthetically pleasing relationship -- the utility of a gamble is the ratio of the weighted expected utility of the gamble and the expected weighting function. Rank dependent utility is characterized by (Camerer, 1989)

$$(13) \qquad V(L) = w(p_1)u(x_1) + (1 - w(p_1))u(x_2),$$

the utility of a gamble is the subjectively weighted sum of the payoffs. So, (13) is similar to (1) except that cumulative probabilities are transformed by a subjective weighting function that is specified such that monotonicity and stochastic dominance are preserved.
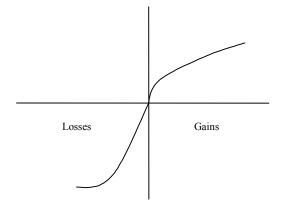
3.5 Prospect Theory

Weighted expected utility and rank dependent utility are designed to accommodate empirical deviations from expected utility without sacrificing expected utility's desirable properties. Agents have underlying preference functions and behave as if they are trying to maximize the functions. Prospect theory was initially developed by two psychologists, Kahneman and Tversky (1979), to be consistent with human thought processes. Prospect theory applies only to gambles for which there are two possible outcomes. This fact makes prospect theory impossible to apply to situations where there are many possible outcomes. Cumulative prospect theory is an extension of prospect theory that is suited for analyzing hedging.

Prospect theory is a procedural theory in which agents make decisions in a two-stage process. In the first stage the agent studies the various lotteries and edits them using decision heuristics. Supporters of prospect theory believe this process to be consistent with how agents actually behave. Instead of using all available information, which maybe overwhelming, agents use "tried and true" decision rules to help them sort out the various states of nature. In the second stage the agent behaves as if the edited lotteries enter a preference function. Stage two is similar to the more traditional approaches but stage one has no counterpart in expected, weighted, or rank dependent utility.

In prospect theory gains and loses are measured according to a reference point. Within the context of hedging the reference point might be current wealth. So the agent is not so much concerned with the payoff of a gamble but the deviation of the gamble from the reference point. The use of a reference point gives rise to another unique feature of prospect theory, the shape of the valuation function. Below the reference point are the losses where the shape of the function is typically convex. In the gains region above the reference point, the function is concave. At the reference point the function is kinked. Another key feature of the valuation function is that it is steeper in the convex (loss) portion of the domain. Figure 9 depicts the shape of a valuation function (Starmer, 2000).
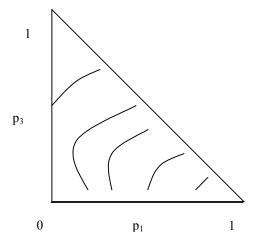
Figure 9: Prospect Theory Valuation Function

Tversky and Kahneman (1992) argue that the shape of the valuation function depicted in figure 8 has two desirable properties with respect to observed human behavior - diminishing sensitivity and loss aversion. If an agent's behavior exhibits diminishing sensitivity then marginal changes relatively farther from the reference point, or kink, are psychologically less important than marginal changes near the reference point. Diminishing sensitivity implies diminishing marginal utility right of the kink and diminishing marginal disutility left of the kink (Starmer, 2000). Agents that exhibit loss aversion will not be attracted to gambles that have a 50 percent change of winning $x and a 50 percent chance of losing $x.

The probabilities are transformed into decision weights in a nonlinear manner. Kahneman and Tversky suggest an increasing $w(p_i)$, sub-additive, and discontinuous at the end points. A prospect theory triangle diagram is included in Figure 10 taken from Camerer (1989).

Figure 10: Prospect Theory Triangle Diagram

The prospect theory diagram is quite exotic compared to the ones considered thus far. The exotic nature is due to the kinked nature of the prospect theory valuation function and because the weighting function is not continuous at the end points. Despite the peculiar look of the figure it is capable of explaining Allais paradox violations of expected utility.

The editing phase is what really differentiates prospect theory from the alternatives to expected utility discussed so far. The editing phase can include different heuristics by the decision maker. For one thing, the agent must decide what the reference point is and then analyze the possible outcomes and conclude, based on the reference point, which outcomes are gains and which are losses.   There is also a dominance heuristic that requires the agent to eliminate stochastically dominated outcomes if they are detected by

the agent. It is possible that the gambles are sufficiently complicated making it difficult to edit out all dominated prospects, so it is possible for dominated prospects to survive the editing phase.

Because the preference function is not linear and there is a possibility that dominated outcomes will not be edited out it is possible that transitivity will be violated. The violations would be indirect as follows: L_1 L_2, L_2 L_3, but L_3 L_1. So, in general, prospect theory does not imply transitive or monotonic preferences.

The editing phase makes it difficult to concisely summarize prospect theory in a convenient manner comparable to equations (1), (5), and (13).

3.6 Cumulative Prospect Theory

Cumulative prospect theory (Tversky and Kahneman, 1992) can be loosely thought of as a combination of rank dependent utility and prospect theory. The evolution from prospect theory to cumulative prospect theory is interwoven with the development of rank dependent utility.   Quiggin was impressed with prospect theory but thought it too limiting, he did not like the potential intransitivities implied by prospect theory, so he went on to develop rank dependent utility. Tversky and Kahneman were impressed with rank dependent utility but they preferred a model with payoffs that are measured as deviations from a reference point, so they developed cumulative prospect theory.

In cumulative prospect theory there is no editing phase and there can be any number out outcomes in the lotteries. Another advantage of cumulative prospect theory over regular prospect theory is that transitivity and monotonicity are maintained thus assuring stochastic dominance preference. The important aspects of prospect theory that are retained include the measurement of payoffs relative to a reference point and consistency with diminishing sensitivity and loss aversion. (Please note that the definition of "sensitivity" is similar but not the same as the definition of sensitivity in weighted expected utility theory.)

The decision weighting function is different for gains then it is for losses in the model summarized below. The superscripts + and - reflect gains and losses. So $\{x_i^+\}$ is equal to $\{x_i\}$ for values above the reference point and 0 otherwise and $\{x_i^-\}$ consists of $\{x_i\}$ values below the reference point and 0 otherwise. The + and – superscripts can also be applied to functions. A function $f^+(x)$ is evaluated over the range of $\{x_i^+\}$, defined above.   Within each category, gain or loss, the prospects are sorted into ascending order. The functions have the following form,

$$(14) \quad V(x) = V(x^+) + V(x^-)$$

$$(15) \quad V(x^+) = \sum_{i=0}^{n} w_i^+(p_i)v(x_i)$$

$$(16) \quad V(x^-) = \sum_{i=-m}^{0} w_i^-(p_i)v(x_i)$$

where x_i are the payoffs relative to the reference point. One other caveat is that $v(.)$ is strictly increasing and $v(0)=0$. The prospects from 0 to n are the gains and the prospects from -m to 0 are the losses. The weighting functions are as follows,

$$(17) \quad w_n^+ = \pi^+(p_n)$$

$$(18) \quad w_{-m}^- = \pi^-(p_{-m})$$

$$(19) \quad w_i^+ = \pi^+(p_i + ... + p_n) - \pi^+(p_{i+1} + ... + p_n), 0 \le i \le i-1$$

$$(20) \quad w_i^- = \pi^-(p_{-m} + ... + p_i) - \pi^-(p_{-m} + ... + p_{i-1}), 1-m \le i \le 0$$

where $\pi(.)$ is the nonlinear probability transformation function that is strictly increasing and maps [0,1] to [0,1]. Furthermore $\pi(0)=0$ and $\pi(1)=1$. Equations (17) to (20) capture the cumulative distribution transformation process used in rank dependent utility adjusted for prospect theory. The adjustment explicitly takes into account the decision maker's differing psychological interpretations of gains and losses.

Tversky and Kahneman recommend, in addition to the limitations mentioned above, that the function $v(.)$ be chosen to yield diminishing sensitivity and loss aversion (1992). To ensure diminishing sensitivity, $v(.)$ is concave above the reference point, $v''(x) \le 0$ when $x \ge 0$, and convex below the reference point, $v''(x) \ge 0$ when $x < 0$. Loss aversion is ensured by requiring that $v'(x) < v'(-x)$ for $x \ge 0$. Loss aversion is a characteristic that Rabin and Thaler (2001) argue is essential for realistically specifying agent risk aversion. Collectively these requirements greatly limit the choices of $v(x)$; for instance, many of the functional forms used in expected utility such as negative exponential do not meet these requirements. The choice of $v(.)$ will be discussed in greater detail in the next section. Diminishing sensitivity is also affected by the choice of $\pi(.)$. Tversky and Kahneman, (1992) require that $\pi(.)$ be concave near zero and convex near 1. This can be accomplished by using an S shaped transformation function.

With respect to risk how are these requirements interpreted? In cumulative prospect theory the agents' attitudes towards risk are more deliberately built into the models. Cumulative prospect theory was conceived in terms of diminishing sensitivity and loss aversion, which are fairly particular requirements. So, to a certain extent there is less flexibility in specifying attitudes towards risk. In this sense cumulative prospect theory is more deliberate compared to the other specifications; it is designed to fit what its creators have found to be typical agent behavior.

Summarizing equations (14) to (20), the value of cumulative prospects is equivalent to the rank dependent utility of the gains using the gain probability weighting function plus the rank dependent utility of the losses using the loss probability weighting function. Cumulative prospect theory can then be summarized as,

$$(21) \qquad u(L) = [w^-(p_1)u(x_1^-) + (1 - w^-(p_1))u(x_2^-)] + [w^+(p_1)u(x_1^+) + (1 - w^+(p_1))u(x_2^+)]$$

where x_i^- and x_i^+ are defined as before.

The axiomization of cumulative prospect theory relevant to this research can be found in Chateauneuf and Wakker (1999). The axioms include, weak ordering, continuity, stochastic dominance preference, and tradeoff consistency. The tradeoff consistency axiom is the most distinguishing. It is similar to the co-monotonic independence axiom of rank dependent utility. Basically it states that gains should be co-monotonic independent from one another and losses should be co-monotonic independent of one another.

Cumulative prospect theory and rank dependent utility are very similar, indeed, it can be shown that, mathematically, rank dependent utility is a special case of prospect theory. If the following relation is true,

$$(22) \qquad \pi^-(p) = 1 - \pi^+(1 - p)$$

cumulative prospect theory simplifies to rank dependent utility. So when π^- is the dual of π^+ cumulative prospect theory is equivalent to rank dependent utility (Chateauneuf and Wakker, 1999). Although, mathematically, rank dependent utility is a special case of prospect theory, qualitatively it seems the other way around. Rank dependent utility does not require diminishing sensitivity or loss aversion, so in a sense, it is a more flexible specification. Hence one could argue that, intuitively, cumulative prospect theory is a special case of rank dependent utility, one that restricts v(.) and π(.) to support diminishing sensitivity and loss aversion.

3.7 Summary of the Alternatives

Weighted expected utility, rank dependent utility, and cumulative prospect theory are three prominent alternatives to expected utility. Each is capable of accommodating the common consequence effect and thus able to explain the Allais paradox. Although the three non-expected utility theories are very different in some respects they have at least one feature in common: the probabilities of the various outcomes enter the utility function in a nonlinear manner.

Weighted expected utility and rank dependent utility are axiomatically based on weakening the independence axiom of expected utility. Prospect theory is quite different it is based on the presumption that utility is best modeled in a two stage process, an editing phase and then an optimization phase that is similar to the basic economic

approach. The crucial difference is that in the expected utility type approaches we economists say "agents behave *as if* such an underlying model exists." The prospect theory approach is more hands-on; they, the psychologists, are trying to develop a model that is more consistent with how agents actually do behave, hence the perceived need for editing. This difference is also apparent in the requirement that agents exhibit diminishing sensitivity and loss aversion.

The waters are muddier with cumulative prospect theory because it does not have an editing phase and looks pretty much like the expected utility type models. The only crucial difference is the payoffs, which in prospect theory and cumulative prospect theory are measured as deviations from some status quo reference point. The founders of cumulative prospect theory also built into the models diminishing sensitivity and loss aversion.

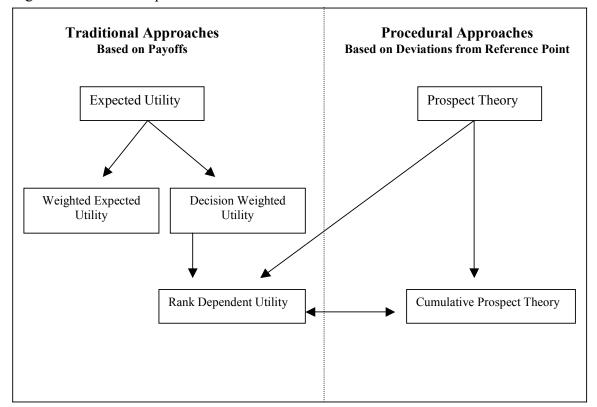The following figure is intended to summarize how the various theories fit together.

Figure 10: Relationships Between Theories

| Traditional Approaches | Procedural Approaches |
| Based on Payoffs | Based on Deviations from Reference Point |

Expected Utility

Prospect Theory

Weighted Expected Utility

Decision Weighted Utility

Rank Dependent Utility

Cumulative Prospect Theory

The left side of the figure depicts theories based on gambles and are, for the most part, developed by economists. The common ancestor to all of these candidates is expected utility; each new theory is designed to accommodate violations of expected utility. The theories to the right include the prospect theory models, which use deviations from a reference point as the relevant payoff. Starmer (2000) refers to these theories as procedural because prospect theory includes an editing phase. However, cumulative

prospect theory does not include an editing phase so it is a bit misleading to refer to it as a procedural theory. The construction of the diagram is somewhat arbitrary in that there are numerous possible constructions depicting the links between the theories. This construction is favored because the branches end with the theories that will ultimately be used in this paper's analysis; weighted expected utility, rank dependent utility, and cumulative prospect theory.

Weighted expected utility is a viable choice because it is a relatively simple generalization of expected utility that yields indifference curves that fan out. Fanning out appears to a great many violations of expected utility, including the common consequence effect. Furthermore, weighted expected utility maintains transitive preferences and stochastic dominance preference, some of expected utility's desirable characteristics. Compared to the alternatives, weighted expected utility is tractable and easy to work with. Starmer (2000) argues that the major problem with weighted expected utility is that the weak independence axiom is un-intuitive, designed only to generate a model with fanning out. This is a plausible argument, requiring that the indifference curves intersect South-West of the triangle diagram is entirely arbitrary. Nevertheless, weighted expected utility's attributes are attractive enough to warrant its consideration.

Rank dependent utility and prospect theory are very similar and considered the leading alternatives to expected utility (Starmer, 2000). The biggest drawback to rank dependent utility and cumulative prospect theory is that they are far more complicated to implement empirically than expected utility and weighted expected utility. Although the analysis is not complete yet, the models will in all likelihood take days to optimize in Matlab.

References

Allais, M. (1953). "Le Comportment de L'homme Rationel Devant le Risque, Critique des Postulates et Axiomes de L'ecole Americane**."** *Econometrica* 21.

Allais, M. (1979). "The Foundations of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School," *Expected Utility Hypotheses and the Allais Paradox.* M. Allais and O. Hagen editors. Reidel.

Arrow, K. (1971). "The Theory of Risk Bearing," *Essays in the Theory of Risk Bearing*. Markham.

Battalio, R., J. Kagel, and D. MacDonald, (1985). "Animals' Choices over Uncertain Outcomes," *American Economic Review* Volume 75.

Camerer, C. (1989). "An Experimental Test of Several Generalized Utility Theories," *Journal of Risk and Uncertainty* Volume 2.

Chateauneuf, A. and P. Wakker (1999). "An Axiomization of Cumulative Prospect Theory for Decision Under Risk." *Journal of Risk and Uncertainty* Volume 18 Number 2.

Chew, S. and K. MacCrimmon (1979a). "Alpha-nu Choice Theory: a Generalization of Expected Utility Theory," working paper 669, University of British Columbia.

Chew, S. and K. MacCrimmon (1979b). "Alpha Utility Theory, Lottery Composition and the Allais Paradox," working paper 686, University of British Columbia.

Chew, S., and W. Waller (1986). "Empirical Tests of Weighted Expected Utility Theory," *Journal of Mathematical Psychology* Volume 30.

Chew, S. (1983). "A generalization of quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox," *Econometrica* Volume 51.

Edwards, W. (1955). "The Prediction of Decisions among Bets," *Journal of Experimental Psychology* Volume 50.

Edwards, W. (1962). "Subjective Probabilities Inferred from Decisions," *Psychology Review* Volume 69.

Ellsberg, D. (1961). "Risk, ambiguity and the Savage axioms," *Quarterly Journal of Economics* Volume 75 Number 4.

Fishburn, P. (1978). "On Handa's 'New Theory of Cardinal Utility' and the Maximization of Expected Return," *Journal of Political Economy* Volume 86.

Machina, M. (1982). "'Expected Utility' Analysis without the Independence Axiom," *Econometrica* Volume 50 Issue 2.

Machina, M. (1987)."Choice Under Uncertainty: Problems Solved and Unsolved," *Economic Perspectives* Volume 1, Number 1.

Marschak, J. (1950). "Rational Behavior, Uncertain Prospects, and Measurable Utility," *Econometrica* Vol 18.

Pratt, J. (1964). "Risk Aversion in the Small and in the Large," *Econometrica* Volume 15.

Prelec, D. (1998). "The Probability Weighting Function," *Econometrica* Volume 66.

Quiggin, J. (1982). "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization* Volume 3 Number 4.

Quiggin, J. (1993). *Generalized Expected Utility Theory: The Rank Dependent Model.* Kluwer Academic Publishers.

Rabin, M. and R. Thaler. (2001) "Anomalies: Risk Aversion," *Journal of Economic Perspectives* Volume 15, Number 1.

Rabin, M. (2000). "Risk Aversion and Expected Utility Theory: A Calibration Theorem," *Econometrica* Volume 65, Number 5.

Starmer, C. (2000). "Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk," *Journal of Economic Literature* Volume 33.

Tversky, A. and D. Kahneman (1992). "Advance in Prospect Theory: Cumulative Representations of Uncertainty," *Journal of Risk and Uncertainty* Volume 5 Number 4.

von Neumann, J. and O. Morgenstern (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

Wakker, P., I. Erev, and U. Weber. (1994). "Comonotonic Independence: The Critical Test Between Classical and Rank Dependent Utility Theories," *Journal of Risk and Uncertainty* Volume 9.

Wakker, P., R. Thaler, and A. Tversky. (1997). "Probabilistic Insurance," *Journal of Risk and Uncertainty* Volume 15, Number 1.